Quaternionic Modular Symbols in Sage

Sage Days 44

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Basic set up

- Fix a level $N \in \mathbb{Z}$.
- Let $\Gamma_0(N)$ be the classical congruence subgroup,

$$\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N | c \} \, .$$

- $\Delta = \operatorname{Div} \mathbb{P}^1(\mathbb{Q})$. $\Gamma_0(N)$ acts on Δ by f.l.t.'s
- V a right $\Gamma_0(N)$-module.
- Get a right action on $\mathrm{Hom}(\Delta_0, V)$:

$$(f|\gamma)(D) = f(\gamma D)|\gamma.$$

- Interested in V-valued modular symbols:

$$\mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, V).$$

- R.Pollack code $\implies$ compute efficiently with this space.

Enter quaternions (I)

- Fix a factorization $N = pDN^+$, satisfying:
  1. $p \nmid DN^+$, and $\gcd(D, N^+) = 1$.
  2. $D = \ell_1 \cdots \ell_r$ squarefree, and r <u>even</u>.
- Remarks:
  1. Not always possible! (e.g. what if $N = \square$?)
  2. Can set $D = 1$.
- Let $B_{/\mathbb{Q}}$ be the quaternion algebra with discriminant D.
- Fix an embedding $\iota_p \colon B \hookrightarrow M_2(\mathbb{Q}_p)$.
- Fix R maximal order in B such that $\iota_p(R) \subset M_2(\mathbb{Z}_p)$.
- Fix $R_0(N^+) \subseteq R$ (resp. $R_0(pN^+)$) an Eichler order of level N^+ (resp. pN^+) such that $R_0(pN^+) \subset R_0(N^+)$.

Enter quaternions (II)

- Define also

$$\Gamma_0^D(N^+) := R_0(N^+)_1^\times, \quad \Gamma_0^D(pN^+) := R_0(pN^+)_1^\times$$

- $D = 1 \implies \Gamma_0^D(pN^+) = \Gamma_0(N)$.
- Let V be a $\Gamma_0^D(pN^+)$-module (e.g. a $\mathrm{SL}_2(\mathbb{Q}_p)$-module).
- Problem: $\mathrm{Hom}_{\Gamma_0^D(pN^+)}(\Delta_0, V)$ makes no sense.
- Solution: Turn to $\mathrm{H}^1(\Gamma_0^D(pN^+), V)$ instead!

Cohomology

- Step back to to $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$.
- Consider the exact sequence of $\Gamma_0(N)$-modules:

$$0 \longrightarrow \Delta_0 \longrightarrow \Delta \xrightarrow{\text{deg}} \mathbb{Z} \longrightarrow 0.$$

- Apply $\mathrm{Hom}(-, V)$ and taking $\Gamma_0(N)$-cohomology:

$$\mathrm{Hom}_{\Gamma_0(N)}(\Delta, V) \xrightarrow{\iota} \mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, V) \xrightarrow{\delta} \mathrm{H}^1(\Gamma_0(N), V)$$

- The map δ is very explicit:

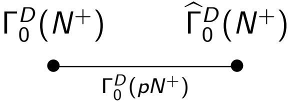
$$(\delta\varphi)_\gamma = \varphi\Big(\{\gamma\infty\} - \{\infty\}\Big),$$

- Also, $\ker(\delta)$ is well understood, since:

$$f \in \mathrm{Hom}_{\Gamma_0(N)}(\Delta, V) \leftrightarrow \{f(c) \colon c \in \Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q})\}.$$

Measures on $\mathbb{P}^1(\mathbb{Q}_p)$ (set $V = \mathbb{Q}$)

- Let $\Gamma = R_0(N^+)[1/p]_1^\times$ (c.f. $\mathrm{SL}(\mathbb{Z}[1/p])$).
- Γ acts (via ι_p) on the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, with fundamental domain:



- Bass-Serre theory $\implies$ $\Gamma = \Gamma_0^D(N^+) \star_{\Gamma_0^D(pN^+)} \Gamma_0^D(N^+)$.
- Shapiro's lemma:

$$\mathrm{H}^1(\Gamma_0^D(pN^+), \mathbb{Q}) = \mathrm{H}^1\left(\Gamma, \mathrm{Hom}(E(\mathcal{T})^o, \mathbb{Q})\right)$$

- Taking Hecke-action into account cuts out:

$$\mathrm{H}^1\left(\Gamma, \mathrm{HC}(\mathbb{Q})\right) \cong \mathrm{H}^1\left(\Gamma, \mathrm{Meas}^0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Q}\right).$$

- Overconvergent methods apply (Pollack-Pollack).
- Application: computing quaternionic Darmon points.

Definite quaternion algebras

- If $B_{/\mathbb{Q}}$ is definite, the corresponding Shimura variety is zero-dimensional.
- Therefore 0^{th} cohomology is interesting!
- We wish to calculate $\mathrm{H}^0\left(\Gamma, \mathrm{Meas}^0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Q})\right)$:
  1. Hecke-module structure.
  2. Integrate functions with respect to one such measure.
  3. Overconvergent methods.
- More generally: $\mathrm{H}^0(\Gamma, \mathrm{HC}(V))$, where V is any Γ-module.

Three projects

1. Stark-Heegner points "à la Darmon-Pollack" for composite conductor ([GM12], w/ Xavier Guitart).
2. Quaternionic p-adic automorphic forms for definite quaternion algebras ([FM12] w/ Cameron Franc).
3. Quaternionic modular symbols for indefinite quaternion algebras (in progress w/ Xavier Guitart).

Stark-Heegner points "à la Darmon-Pollack"

- Ported code from `shp` "external" package to Sage (uses Pollack's OMS code).
- Elementary matrix decompositions ([GM12]) allow us to work with composite level elliptic curves.
- Code base: $\sim$ 850 lines of poorly documented code. ▸ Go .
- Project goal: Get a "Stark-Heegner point calculator".
- $\implies$ Explicit class field theory!

Definite quaternion algebras

- Started from a joint project with C. Franc.
- Main classes:
  1. `BruhatTitsTree`: an implementation of the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, with self-adapting precision.
  2. `BTQuotient`: Computing a fundamental domain of $\mathcal{T}$ for the action of definite quaternionic Γ.
  3. `HarmonicCocycles`: Hecke-module parent/element structure of $\mathrm{H}^0(\Gamma, \mathrm{HC}(V_n))$.
  4. `pAutomorphicForms`: Lift harmonic cocycles to elements of $\mathrm{H}^0(\Gamma, \mathrm{coInd}_{\Gamma_0^D(pN^+)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathcal{V})$.
  5. `OCVn`: Implementation of V_n and $\mathcal{V}_n$, overconvergent and non-overconvergent treated uniformly.
- Code base: $\sim$ 4700 lines of reasonably documented code. Go

Definite quaternion algebras (II)

Project goals:

1. Finish documentation and testing.
2. Remove "external" dependencies:
   1. When defining non-maximal orders.
   2. When finding p-adic splittings.
3. Reuse distributions from modular symbols.
4. Make it interact with elliptic curves.
5. Polish the (already existing) functionality for p-adic Heegner points "à la Greenberg's Thesis".
   $\Longrightarrow$ Heegner point p-adic calculator.

Indefinite quaternion algebras

- Methods arising from work in progress with X. Guitart.
- Problem: finite presentation of $\Gamma_0^D(pN^+)$ and $\Gamma_0^D(N^+)$?
  $\Longrightarrow$ Voight's "external" routines.
- Main classes:
  1. `ArithGroup`: working with $\Gamma_0^D(pN^+)$ or $\Gamma_0^D(N^+)$.
  2. `BigArithGroup`: working with Γ, which is seen as an amalgam $\Gamma = \Gamma_0^D(N^+) \star_{\Gamma_0^D(pN^+)} \Gamma_0^D(N^+)$.
  3. `Cohomology`: Hecke module structure, for now only with trivial coefficients (corresponding to weight 2).
  4. `Homology`: computing with elements of $H_1(\Gamma, \operatorname{Div} \mathcal{H}_p)$.
  5. Natural pairing

$$H^1(\Gamma, \operatorname{Meas}^0(\mathbb{P}^1(\mathbb{Q}_p))) \times H_1(\Gamma, \operatorname{Div} \mathcal{H}_p) \to \mathbb{C}_p$$

- In progress: overconvergent integration.
- Code base: $\sim$ 2000 lines of evolving code. Go

Indefinite quaternion algebras (II)

Project goals:

1. Extensive testing/documentation.
2. Reuse code from Pollack's implementation.
3. Implement higher weight modules.
4. Implement overconvergent algorithm.

   $\Longrightarrow$ Quaternionic Stark-Heegner point calculator.

Conclusion

1. Restricting to matrices is a bad idea.
2. Overconvergent methods yield algorithms for (conjecturally) finding:
    1. Algebraic points on elliptic curves.
    2. Ring class fields.
3. This stuff is <u>not</u> in ~~Magma~~ "the other software", incentive for people to move to Sage.
4. Volunteers in the room to implement quaternion algebras?

Thank you !

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