The Laurent and Robbins phenomena

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$$\cdots \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 3 \quad 7 \quad 23 \quad \cdots$$

Somos-4

$$x_0 = x_1 = x_2 = x_3 = 1$$
$$x_n = (x_{n-1}x_{n-3} + x_{n-2}^2)/x_{n-4}$$

$\ldots, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898, 126742987, 1687054711, 47301104551, 1123424582771, 32606721084786, 1662315215971057, 61958046554226593, 4257998884448335457, \ldots$

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$$x_{18} = \frac{47301104551 \cdot 126742987 + 1687054711^2}{7869898}$$

Integrality

Theorem: x_n is an integer for all $n \in \mathbf{Z}$

Proof:

a. All x_n are nonzero.

b. Define auxiliary quantities y_n and z_n:

$$y_n = (x_n x_{n+3}^2 + x_{n+2}^3)/x_{n+1}, \qquad z_n = (x_n^2 x_{n+4} + x_{n+1}^3)/x_{n+2}$$

c. Prove by simultaneous induction that:

$$\begin{array}{ll} (1) & x_n, x_{n+1}, x_{n+2}, x_{n+3}, y_n, z_n \in \mathbf{Z} \\ (2) & x_n, x_{n+1}, x_{n+2}, x_{n+3} \quad \text{are pairwise coprime} \end{array}$$

Reverend Charles Dodgson

Jacobi's identity

Jacobi (and others): If A is an $n \times n$ matrix over your favorite ring, and $D_{i;j}$ is the determinant of the matrix obtained from A by excising the i-th row and j-th column then:

$$\det(A) \cdot D_{1,n;1,n} = D_{n;n}\, D_{1;1} - D_{1;n}\, D_{n;1}$$

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$$Outer \cdot Inner = NW \cdot SE - NE \cdot SW$$

Proof of Jacobi's identity

a. WLOG, the matrix entries are indeterminates.

b. Adding a multiple of an inner row (resp., column) to any other row (resp., column) leaves all 6 determinants unchanged.

c. WLOG

$$A = \begin{bmatrix} a & 0 & \dots & 0 & b \\ 0 & e_1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & e_{n-2} & 0 \\ c & 0 & \dots & 0 & d \end{bmatrix}$$

and

$$(adE - bcE)E = aE \cdot Ed - bE \cdot Ec$$

6	6	5	1	1
3	4	4	1	1
2	4	6	3	4
2	5	9	6	9
1	3	6	5	9

6		6		5		1		1
	6		4		1		0	
3		4		4		1		1
	4		8		6		1	
2		4		6		3		4
	2		6		9		3	
2		5		9		6		9
	1		3		9		9	
1		3		6		5		9

6		4		1		0
	4		4		1	
4		8		6		1
	4		6		3	
2		6		9		3
	5		9		6	
1		3		9		9

6		4		1		0
	8		4		1	
4		8		6		1
	2		6		3	
2		6		9		3
	0		3		9	
1		3		9		9

8		4		1
	8		6	
2		6		3
	6		9	
0		3		9

8		4		1
	5		1	
2		6		3
	1		5	
0		3		9

5 1

6

1 5

5		1
	4	
1		5

p-adic Integrality

N.B.: p-adics $\equiv$ any DVR

If x_1, x_2, x_3, x_4 are p-adic units, and the Somos-4 recursion never hits zero, then for all n

$$v(x_n) \geq 0$$

i.e., x_n is a p-adic integer, despite the division in the recursion.

Proof: as above.

Laurent phenomenon

Theorem: If x_1, x_2, x_3, x_4 are indeterminates and the Somos-4 recursion is used to calculate

$$x_n = x_n(x_1, x_2, x_3, x_4)$$

then every x_n lies in the Laurent ring

$$\mathbf{Z}[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}, x_4^{\pm}]$$

Proof: As above (except that step a. is unnecessary).

Laurent phenomenon, cont.

Sergey Fomin and Andrei Zelevinsky prove many instances of the Laurent phenomenon in their 2001 paper *The Laurent Phenomenon*, using the **Caterpillar Lemma**

This was one of the origins of the theory of Cluster Algebras

A p-adic "computational framework"

You have a memory with n locations that can store values in a p-adic ring.

Each step of the computation must be of the following form: pick a location with value x and replace the value with a polynomial of the other locations divided by x, i.e.,

$$x_{\text{new}} = P(\text{other locations})/x_{\text{old}}$$

The polynomial P is called an exchange polynomial. We do not allow consecutive steps to target the same location.

p-adic stability

Let c be a positive integer. A computational framework as above satisfies the "p-adic Robbins-c property" (or is p-adic Robbins-c stable, or is an instance of the p-adic Robbins-c phenomenon) if

$$v(x - x') \geq r - c \cdot e$$

where

- x is the result of the p-adic computation,
- x' is the result of some p-adic computation with relative precision r,
- e is the maximum valuation of the divisors in the computation, and
- we assume that $c \cdot e \leq r$.

Somos-4 example

Let x_n be a Somos-4 in the p-adics, and let x'_n be a sequence with the the same four initial terms, but with all calculations done with with relative p-adic precision r. Define

$$e := \max_{4 \le k \le n-4} v(x_k)$$

Theorem: If $e < r$, then

$$v(x_n - x'_n) \ge r - e\ .$$

In other words, Somos-4 is p-adic Robbins stable.
Robbins conjectured that the Dodgson recursion also has this property.

Cluster Algebras

A (skew-symmetric, geometric) cluster algebra (without coefficients) starts with

- n memory locations, initialized with indeterminates, and
- a skew-symmetric integer matrix M.

A cluster computation (a) changes a given x_k by “binomial” exchange polynomials

$$x_{k,\mathrm{old}} x_{k,\mathrm{new}} = [x^{M(k)}]_+ + [x^{-M(k)}]_+$$

where $M(k)$ is the k-th row of M (so that $row(k)_k = 0$) and

$$[x^v]_+ = \prod_{v_i > 0} x_i^{v_i}$$

and (b) mutates the matrix M according to a specific rule.

We conjecture that cluster algebra computations are p-adically Robbins stable.

Algebraic Formulation of p-adic stability

A (re)formulation of relative p-adic precision r:

> *An adversary/devil/pixie is allowed to secretly multiply any stored value x by an r-unit, i.e., an element of the form $1 + \varepsilon$ where $v(\varepsilon) \geq r$.*

The Somos-4 recursion with errors is then

$$x_n x_{n-4} = (1 + \varepsilon_{0,n}) x_{n-1} x_{n-3} + (1 + \varepsilon_{1,n}) x_{n+2}^2.$$

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Now let the $\varepsilon_{i,n}$ be indeterminates!

Theorem: If x_n is computed with the Somos-4 recursion with errors then x_n lies in the ring

$$\mathbf{Z}\left[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}, x_4^{\pm}, \frac{\varepsilon_{i,j,k}}{x_k}\right]$$

where the (i,j,k) range over the set $\{(i,j,k) : i = 0, 1, \quad |j-k| \leq 1, \quad 4 \leq k \leq n-4\}$.

We say that Somos-4 satisfies the Robbins phenomenon.

The proof of the above theorem is motivated by the earlier proof, but is significantly more elaborate.

A computational framework satisfies the (algebraic) *Robbins phenomenon* if each final memory location lies in an analogous enlarged Laurent ring.

We conjecture that frameworks coming from cluster algebras satisfy this property.

Examples

- $x_0 x_4 = x_1 x_3 + x_2^2$ is provably Laurent and Robbins
- $x_0 x_5 = x_1 x_4 + x_2 x_3$ is Laurent (though we know of no easy proof) and Robbins
- $x_0 x_6 = x_1 x_5 + x_2 x_4$ is Laurent and (experimentally) Robbins
- $x_0 x_7 = x_1 x_6 + x_2 x_5$ ditto
- $x_0 x_6 = x_1 x_5 + x_3^2 + x_2 x_4$ is Laurent and **not** Robbins, but it is (experimentally) Robbins-2 and (provably) Robbins-5
- The Dodgson recurrence is (experimentally) Robbins, and (provably) Robbins-3

Examples, cont.

p prime, $k \geq 3$ and $c \in \mathbf{Z}$. Consider

$$x_k x_0 = x_1^2 + \cdots + x_{k-1}^2 + c \sum_{0<i<j<k} x_i x_j \; .$$

over $\mathbf{Z}_p$ This is Laurent. Let *NR* denote the assertion that $c^2 - 4$ is a quadratic nonresidue mod p. Experimental results for $k = 4$:

- Robbins-2 for all p, and $|c| < 6$.
- For $c = 0$ and $c = 2$, Robbins for all p.
- For $c = 1$, $c = \pm 3$, and $c = \pm 4$, Robbins for $p = 2$ or *NR*.
- For $c = \pm 5$, Robbins for $p = 2, 5$, or *NR*.

For fixed c, it seems that the correction factor can grow roughly linearly with p.