Self-paced

# Student Study Modules

for

# Calculus I–Calculus III

## Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

## Derivatives of polynomials

In this module we will determine more efficient ways of computing the derivatives of *polynomials*. We use the precise definition of the derivative to explain these methods.



## Defining the problem

Recall the definition of the derivative as a function.

**Definition:** *(The derivative as a function)*
If a function $f$ is differentiable at every point in its domain, then *the derivative of $f$* is
$$f'(x) = \lim_{x\to a} \frac{f(x+h)-f(x)}{h}.$$

It is tedious to compute the derivative using the precise definition. Polynomials have a certain structure that is different from trigonometric, exponential, or logarithmic functions. We'd like to exploit that structure and find a shorter way of computing the derivative for polynomials.

# SAGE worksheets

You will need a blank SAGE worksheet for this module.



# The derivative of a polynomial

Let $f(x)$ be a polynomial. We want to compute the derivative of $f(x)$.



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Let $f(x)$ be a polynomial. We want to compute the derivative of $f(x)$.

The simplest form of a polynomial is a monomial $x^n$, where $n$ is a positive integer. Before looking at this "generic" monomial, we'll look at a few examples: $x$, $x^2$, and $x^3$.



# The derivative of $x$

Using the precise definition, the derivative of $f(x) = x$ is

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(\cancel{x}+h)-\cancel{x}}{h} = \lim_{h\to 0} 1.$$

Hence we can say

**Theorem:**

- If $f(x) = x$, then $f'(x) = 1$.

# The derivative of $x^2$

Using the precise definition, the derivative of $f(x) = x^2$ is

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$



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$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(\cancel{x^2} + 2xh + h^2) - \cancel{x^2}}{h}\end{aligned}$$



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(Factored a common $h$ in the numerator.)



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$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\ &= \lim_{h\to 0} 2x + \cancelto{0}{h}.\end{aligned}$$

(Evaluated the limit be substituting $h = 0$.
We can do this because $2x + h$ is a continuous function.)

# The derivative of $x^2$

Using the precise definition, the derivative of $f(x)=x^2$ is

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}2x+\cancel{h}^{\,0}$$
$$=2x.$$

Hence we can say

**Theorem:**

- If $f(x)=x$, then $f'(x)=1$.
- If $f(x)=x^2$, then $f'(x)=2x$.



# The derivative of $x^3$

Using the precise definition, the derivative of $f(x)=x^3$ is

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$



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$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$
$$=\lim_{h\to 0}\frac{(\cancel{x^3}+3x^2h+3xh^2+h^3)-\cancel{x^3}}{h}$$



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$$=\lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}}$$

(Factored a common $h$ in the numerator.)

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(Evaluated the limit be substituting $h = 0$.
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Using the precise definition, the derivative of $f(x) = x^3$ is

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 3x^2 + 3x\cancelto{0}{h} + \cancelto{0}{h^2}$$
$$= 3x^2.$$

Hence we can say

**Theorem:**

- If $f(x) = x$, then $f'(x) = 1$.
- If $f(x) = x^2$, then $f'(x) = 2x$.
- If $f(x) = x^3$, then $f'(x) = 3x^2$.



## Geometrically

You can see this geometrically in SAGE. Plot $f(x)$ and $f'(x)$ on the same graph, and observe how the graph of $f(x)$ relates to the graph of $f'(x)$.



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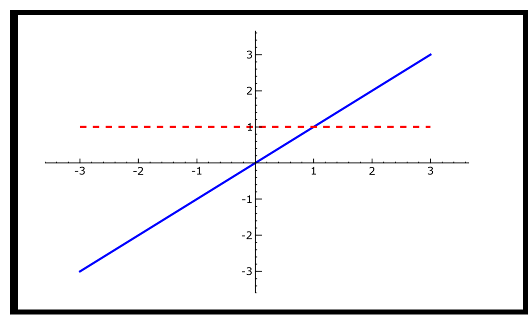
When $f(x) = x$, $f'(x) = 1$. This reflects the constant slope of the tangent line.



## Geometrically

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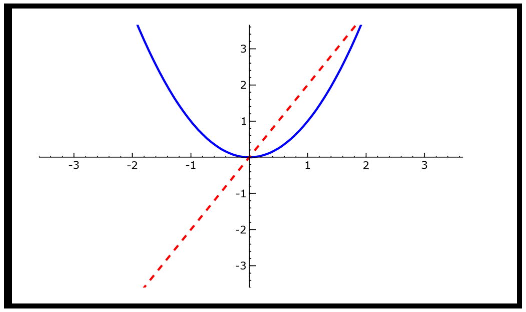
When $f(x)=x^2$, $f'(x)=2x$.



Notice that when the slope of the line tangent to $x^2$ is negative—as it is for $x \in (-\infty, 0)$—the $y$-values of $f'(x)$ are negative.



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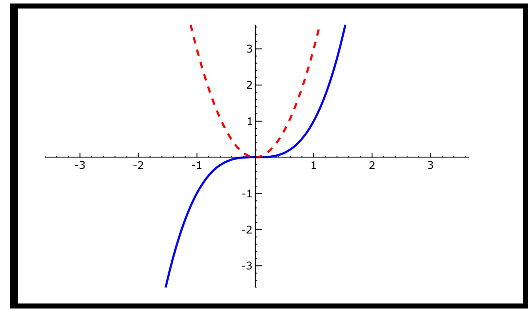
You can see this geometrically in SAGE. Plot $f(x)$ and $f'(x)$ on the same graph, and observe how the graph of $f(x)$ relates to the graph of $f'(x)$.

When $f(x)=x^3$, $f'(x)=3x^2$.



Notice that the $y$-values of $f'(x)$ are always positive in this graph, reflecting the fact that the slope of the line tangent to $f(x)$ is always positive.



## Did you notice a pattern?

You may have noticed the outlines of a pattern here:

| $f(x)$ | $f'(x)$ |
|---|---|
| $x$ | $1=1x^0$ |
| $x^2$ | $2x$ |
| $x^3$ | $3x^2$ |
| $x^4$ | ?? |

Can you guess what the next derivative should be?



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| $x^3$ | $3x^2$ |
| $x^4$ | $4x^3$ |

Can you guess what the next derivative should be?

It looks as if the derivative of $f(x)=x^4$ should be $f'(x)=4x^3$. You might guess further that the derivative of $f(x)=x^n$ is $f'(x)=nx^{n-1}$.

# A pattern in the precise definition

To prove this guess, look back at the algebra of the precise definition of the derivative.

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A pattern in the precise definition

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Everything is the same each time except

$$
\begin{aligned}
(x+h) &= x^1 + h \\
(x+h)^2 &= x^2 + 2xh + h^2 \\
(x+h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3
\end{aligned}
$$

Notice a pattern here: $(x+h)^n = x^n + h(nx^{n-1} + g)$ where $g$ is a polynomial with a power of $h$ *in every term*.

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Notice a pattern here: $(x+h)^n = x^n + h(nx^{n-1} + g)$ where $g$ is a polynomial with a power of $h$ *in every term*.

Test this pattern using SAGE. For different values of $n$, expand $(x+h)^n$ and verify that the result has the form $x^n + h(nx^{n-1} + g)$ as explained above.

(Hint 1: Use the `expand` command to expand each product. For example, the command `expand((x+h)^4)` will expand the product $(x+h)^4$.)

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(Hint 2: You can see as many examples as you want using a `for` loop. For example,

```
var('h')
for n in range(19):
  expand((x+h)^(n+2))
```

will compute $(x+h)^2$, $(x+h)^3$, … $(x+h)^{20}$.)

# A pattern in the precise definition

To prove this guess, look back at the algebra of the precise definition of the derivative.

Everything is the same each time except

$$\begin{aligned}(x+h) &= x^1+h\\(x+h)^2 &= x^2+2xh+h^2\\(x+h)^3 &= x^3+3x^2h+3xh^2+h^3\end{aligned}$$

Notice a pattern here: $(x+h)^n = x^n + h(nx^{n-1}+g)$ where $g$ is a polynomial with a power of $h$ *in every term*.

Test this pattern using SAGE. For different values of $n$, expand $(x+h)^n$ and verify that the result has the form $x^n + h(nx^{n-1}+g)$ as explained above.

Try to think about why this pattern is true for every positive integer $n$. We will not dwell on a proof, but your instructor and/or textbook can provide one if you ask.



# The derivative of a monomial

We can now prove

**Theorem:** *(The derivative of a monomial)*
If

- $f(x) = x^n$ for some $n \in \mathbb{N}$,

then

- $f'(x) = nx^{n-1}$.



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Why?

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

( From the precise definition of a derivative. )



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( From the pattern described previously. )

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Why?

$$f'(x) = \lim_{h\to 0} \frac{(\cancel{x^n} + h(nx^{n-1} + g) - \cancel{x^n}}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(nx^{n-1} + g)}{\cancel{h}}$$

( It is important to note that since $h \to 0$ we can assume that $h \neq 0$.
Hence this division is perfectly legal. )



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Why?

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(nx^{n-1} + g)}{\cancel{h}}$$
$$= \lim_{h\to 0} nx^{n-1} + \cancel{g}^{0}$$
$$= nx^{n-1}.$$

( Because $g$ is a polynomial with a power of $h$ *in every term*, and $h \to 0$. )



## Example

We can now say, *without using the precise definition*, that

- if $f(x) = x^{27}$,
- then $f'(x) = 27x^{26}$.

That's a *lot* easier than using the precise definition!



## Other forms of $x^n$

We have shown that the derivative of $x^n$ is $nx^{n-1}$ when $n$ *is a positive integer*. What about cases where $n$ is a negative integer, or a fraction, or *any* real number?

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The same pattern extends to all *nonzero* real numbers $n$. Thus we can say that

| $f(x)$ | $f'(x)$ | reasoning |
|---|---|---|
| $x^\pi$ | $\pi x^{\pi-1}$ | shortcut |
| $\frac{1}{x}$ | $-\frac{1}{x^2}$ | $\frac{1}{x}=x^{-1}$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ | $\sqrt{x}=x^{\frac{1}{2}}$ |

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We *do not* say that the derivative of $x^0$ is $0x^{-1}$, because $x^0=1$, and the derivative of a constant is $0$. So the property of the derivative of $x^n$ does not work for every real number.



## The derivative of a polynomial

Using the properties of derivatives, we can extend this shortcut for *monomials* to a shortcut for *polynomials*. For example, suppose that $f(x)=4x^2+3x+2$.

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- The shortcut tells us that the derivative of $x^2$ is $2x$, and the derivative of $x$ is 1.

$$4\cancelto{2x}{(x^2)}+3\cancelto{1}{(x)}+2.$$

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- The shortcut tells us that the derivative of $x^2$ is $2x$, and the derivative of $x$ is 1.
- One property of the derivative tells us that the derivative of the last term, 2, is 0.

$$4\left(\cancel{x^2}\right)^{2x} + 3\left(\cancel{x}\right)^{1} + \cancel{2}^{0}.$$

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- One property of the derivative tells us that the derivative of the last term, 2, is 0.
- Another property of the derivative tells us that the derivative of $4x^2$ is 4 times the derivative of $x^2$, or $4 \cdot 2x = 8x$.

$$\cancel{\left(4\left(\cancel{x^2}\right)^{2x}\right)}^{8x} + 3\left(\cancel{x}\right)^{1} + \cancel{2}^{0}.$$

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- Likewise, the derivative of $3x$ is $3 \cdot 1 = 3$.

$$\cancel{\left(4\left(\cancel{x^2}\right)^{2x}\right)}^{8x} + \cancel{\left(3\left(\cancel{x}\right)^{\cancel{1}}\right)}^{3} + \cancel{2}^{0}.$$

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- Likewise, the derivative of $3x$ is $3 \cdot 1 = 3$.
- Another property of the derivative tells us that *the derivative of a sum* is *the sum of the derivatives*.

$$\cancel{\left(\cancel{\left(4\left(\cancel{x^2}\right)^{2x}\right)}^{8x} + \cancel{\left(3\left(\cancel{x}\right)^{\cancel{1}}\right)}^{3} + \cancel{2}^{0}\right)}^{8x+3+0}.$$

## The derivative of a polynomial

Using the properties of derivatives, we can extend this shortcut for *monomials* to a shortcut for *polynomials*. For example, suppose that $f(x)=4x^2+3x+2$.

- The shortcut tells us that the derivative of $x^2$ is $2x$, and the derivative of $x$ is 1.
- One property of the derivative tells us that the derivative of the last term, 2, is 0.
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- Likewise, the derivative of $3x$ is $3\cdot 1=3$.
- Another property of the derivative tells us that *the derivative of a sum* is *the sum of the derivatives*.

$$\cancelto{8x+3+0}{\left(\cancelto{8x}{\left(4\cancelto{2x}{\left(x^2\right)}\right)}+\cancelto{3}{\left(3\cancelto{1}{(x)}\right)}+\cancelto{0}{2}\right)}.$$

Thus $f'(x)=8x+3$.



## Conclusion

- We found a shortcut for the derivative of a monomial: if $f(x)=x^n$ then $f'(x)=nx^{n-1}$.
- This shortcut works for all $n\in\mathbb{R}\backslash\{0\}$.
- To find this shortcut, we used the precise definition of the derivative and exploited some patterns that we noticed for $n=1,2,3$.
- We can use properties of the derivative to extend this shortcut to all polynomials.



## End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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