Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Overview

Up to now we have only considered limits as they approach *a fixed number a*:

$$\lim_{x\to a} f(x).$$

Often we are interested as well in the *long-term* behavior of a function:

Question:

- What happens to the y values as x increases without bound? That is, what is
$$\lim_{x\to\infty} f(x)?$$
- What happens to the y values as x decreases without bound? That is, what is
$$\lim_{x\to-\infty} f(x)?$$

Three sample problems

Let

- $f(x)=\dfrac{3x+1}{x}$,
- $s(x)=\sin(x)$, and
- $d(x)=\dfrac{1}{x}$.

We want to estimate, or even evaluate,

$$\lim_{x\to\pm\infty} f(x) \quad\text{and}\quad \lim_{x\to\pm\infty} s(x) \quad\text{and}\quad \lim_{x\to\pm\infty} d(x),$$

assuming they exist!.

The limit at infinity

Recall from previous lessons that:

- the mathematical term *limit* indicates the value a variable or a function approaches;
- we can use numerical or graphical methods to estimate the value of a limit, if it exists; and
- we can use the precise definition of a limit (ε-δ) to show rigorously that our estimate is correct, or that none exists.

Infinity is not a number!

It is important to keep in mind that

infinity is not a number!

We cannot "substitute" it in place of x, add to it, subtract from it, and so forth.

Infinity is a *word* that we use as a shorthand for an *idea*: namely, increasing or decreasing without bound.

SAGE worksheets

In this lab you will not need a special SAGE worksheet, but you will need to have a new SAGE worksheet handy, so go ahead and login to SAGE and open a new worksheet.

First example: numerical

Let's use all three methods on the first example,

$$\lim_{x\to\pm\infty} f(x) \quad \text{where} \quad f(x) = \frac{3x+1}{x}.$$

We will start with a numerical estimate: what happens to the values of f as x increases without bound?

First example: numerical

Let's use all three methods on the first example,

$$\lim_{x\to\pm\infty} f(x) \quad \text{where} \quad f(x)=\frac{3x+1}{x}.$$

We will start with a numerical estimate: what happens to the values of f as x increases without bound?

Since x increases without bound, we'll build a table of x-y values where x increases. Feel free to pick grotesquely big numbers!

x	1	10	100	1000	10000	1,000,000,000
y	?	?	?	?	?	?

First example: numerical

Let's use all three methods on the first example,

$$\lim_{x\to\pm\infty} f(x) \quad \text{where} \quad f(x)=\frac{3x+1}{x}.$$

We will start with a numerical estimate: what happens to the values of f as x increases without bound?

Since x increases without bound, we'll build a table of x-y values where x increases. Feel free to pick grotesquely big numbers!

x	1	10	100	1000	10000	1,000,000,000
y	?	?	?	?	?	?

Remember that you can use SAGE to pass quickly through the computations: define a function f, then evaluate f at each point. You can also use a loop: `[f(a) for a in [1,10,...]]` as described in the module on intuitive limits.

First example: numerical

Let's use all three methods on the first example,

$$\lim_{x\to\pm\infty} f(x) \quad \text{where} \quad f(x)=\frac{3x+1}{x}.$$

We will start with a numerical estimate: what happens to the values of f as x increases without bound?

Since x increases without bound, we'll build a table of x-y values where x increases. Feel free to pick grotesquely big numbers!

x	1.0	10.0	100.0	1000.0	10000.0	1,000,000,000.0	$\to\infty$
y	4.0	3.1	3.01	3.001	3.0001	3.000000001	$\to 3$

If everything goes well, you should see numbers corresponding to the table above. This suggests that

$$\lim_{x\to\infty} f(x)=3.$$

First example: graphical

Let's use all three methods on the first example,

$$\lim_{x\to\pm\infty} f(x) \quad \text{where} \quad f(x)=\frac{3x+1}{x}.$$

Let's try a graphical approach. Plot the graph of $f(x)$ several times, with different maximum values of x.

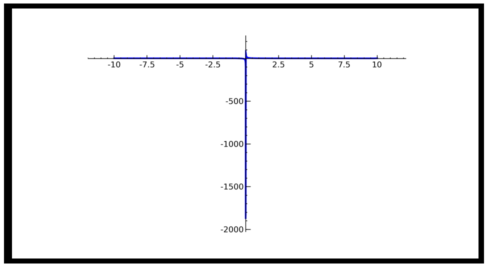
First example: graphical

Let's use all three methods on the first example,

$$\lim_{x\to\pm\infty} f(x) \quad \text{where} \quad f(x)=\frac{3x+1}{x}.$$

Let's try a graphical approach. Plot the graph of $f(x)$ several times, with different maximum values of x.

If we try `plot(f,-10,10)` we see the following (plot may vary):



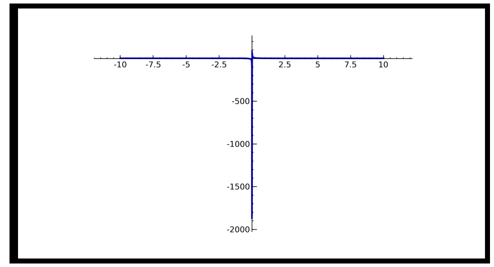
From the graph we might guess that

$$\lim_{x\to\infty} f(x) = 0.$$

But this disagrees with our numerical estimate!

What happened?

Take a closer look at the graph:



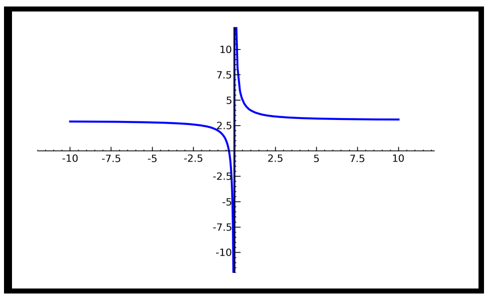
Do you see how the y values have such large magnitude? They go up to 100-ish and down to -1900-ish.

To get a more accurate picture, we can use the `show()` command with the `ymin` and `ymax` attributes. Try typing this:

```
fplot = plot(f,-10,10)
fplot.show(ymin=-10,ymax=10)
```


First example: graphical

This new approach gives us a much better picture:



This new plot *suggests* that the y values approach 3.

First example: graphical

However, our graph isn't that convincing. It *suggests* that the y values approach 3. But we're only going out to $x=10$. Lots of funny things can happen beyond $x=10$.

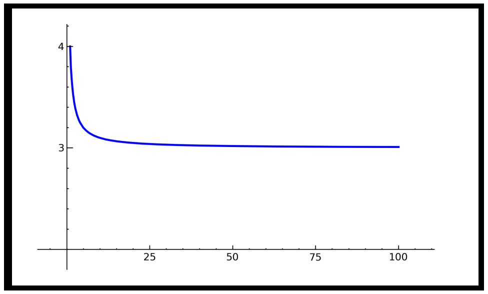
To get around this, experiment with the x-interval in the plot.

First example: graphical

However, our graph isn't that convincing. It *suggests* that the y values approach 3. But we're only going out to $x = 10$. Lots of funny things can happen beyond $x = 10$.

To get around this, experiment with the x-interval in the plot.

We might, for example, try the interval $[1, 100]$:

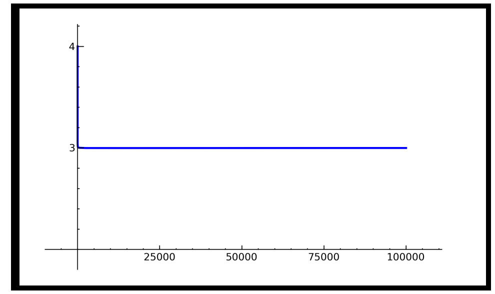


First example: graphical

However, our graph isn't that convincing. It *suggests* that the y values approach 3. But we're only going out to $x = 10$. Lots of funny things can happen beyond $x = 10$.

To get around this, experiment with the x-interval in the plot.

Or the interval $[1, 10000]$:

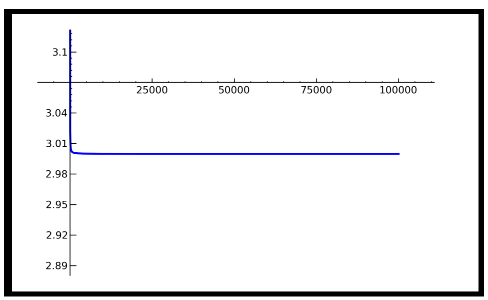


First example: graphical

However, our graph isn't that convincing. It *suggests* that the y values approach 3. But we're only going out to $x = 10$. Lots of funny things can happen beyond $x = 10$.

To get around this, experiment with the x-interval in the plot.

You can also shrink the y-interval in the `show()` command to $[2.9, 3.1]$:



(Use `ymin=2.9,ymax=3.1` as options to `show()`.)

First example: graphical

However, our graph isn't that convincing. It *suggests* that the y values approach 3. But we're only going out to $x = 10$. Lots of funny things can happen beyond $x = 10$.

To get around this, experiment with the x-interval in the plot.

No matter how we change the graph, it *appears* that the y values are approaching $y = 3$ as x increases without bound. Intuitively, we believe that

$$\lim_{x \to \infty} f(x) = 3.$$

(The same also appears to be true for $-\infty$.)

A precise approach!

How can we *convince* ourselves that a limit at infinity is L? We will need a precise definition for the limit at infinity.

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How can we *convince* ourselves that a limit at infinity is L? We will need a precise definition for the limit at infinity.

For any measurable distance ε, we want to know that eventually the y value of $f(x)$ is indistinguishable from the limit 3.

A precise approach!

How can we *convince* ourselves that a limit at infinity is L? We will need a precise definition for the limit at infinity.

In other words, we want

- for all $\varepsilon > 0$,
- sooner or later, *every* x value
- produces a y value that is closer to L than ε.

A precise approach!

How can we *convince* ourselves that a limit at infinity is L? We will need a precise definition for the limit at infinity.

Rethinking our wording again,

Definition: *(Limit at ∞)*
$\lim_{x\to\infty} f(x) = L$ if

- for all $\varepsilon > 0$,
- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

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- for all $\varepsilon > 0$,
- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

Let's try this with the example, using $\varepsilon = 0.01$. We want

$$|f(x) - L| < 0.01.$$

Module: limits at infinity 10

A precise approach!

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$$|f(x) - L| < 0.01$$

$$\left|\frac{3x+1}{x} - 3\right| < 0.01.$$

Module: limits at infinity 10

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Let's try this with the example, using $\varepsilon = 0.01$. We want

$$\left|\frac{3x+1}{x} - 3\right| < 0.01$$

$$-0.01 < \frac{3x+1}{x} - 3 < 0.01.$$

(Property of this absolute value inequality.)

Module: limits at infinity 10

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- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

Let's try this with the example, using $\varepsilon = 0.01$. We want

$$-0.01 < \frac{3x+1}{x} - 3 < 0.01$$

$$3 - 0.01 < \frac{3x+1}{x} < 3 + 0.01.$$

(Added 3 to both sides.)

A precise approach!

How can we *convince* ourselves that a limit at infinity is L? We will need a precise definition for the limit at infinity.

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$\lim_{x\to\infty} f(x) = L$ if

- for all $\varepsilon > 0$,
- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

Let's try this with the example, using $\varepsilon = 0.01$. We want

$$3 - 0.01 < \frac{3x+1}{x} < 3 + 0.01$$
$$x(3-0.01) < 3x+1 < x(3+0.01).$$

(Multiplied x to both sides.
Since $x > 0$ as $x \to \infty$,
the direction of the inequalities does not change.)

A precise approach!

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Rethinking our wording again,

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- $|f(x) - L| < \varepsilon$.

Let's try this with the example, using $\varepsilon = 0.01$. We want

$$x(3-0.01) < 3x+1 < x(3+0.01)$$
$$-0.01x < 1 < 0.01x.$$

(Subtracted $3x$ from both sides.)

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$\lim_{x\to\infty} f(x) = L$ if

- for all $\varepsilon > 0$,
- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

Let's try this with the example, using $\varepsilon = 0.01$. We want

$$-0.01x < 1 < 0.01x$$
$$-0.01x < 1 \text{ and } 1 < 0.01x.$$

(Meaning of the inequality.)

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$\lim_{x\to\infty} f(x) = L$ if

- for all $\varepsilon > 0$,
- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

Let's try this with the example, using $\varepsilon = 0.01$. We want

$$-0.01x < 1 \text{ and } 1 < 0.01x$$
$$\text{so we want} \quad x > -\frac{1}{0.01} \text{ and } x > \frac{1}{0.01}.$$

The larger of these is $1/0.01 = 100$, so we want $x > 100$.

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Let's try this with the example, using $\varepsilon = 0.01$. We want

$$-0.01x < 1 \text{ and } 1 < 0.01x$$

$$\text{so we want} \quad x > -\frac{1}{0.01} \text{ and } x > \frac{1}{0.01}.$$

The larger of these is $1/0.01 = 100$, so we want $x > 100$.

In fact, $f(101) \approx 3.0099$ and

$$|3.0099 - 3| = .0099 < .01.$$

Second example

Let's look at $s(x) = \sin(x)$.

Second example: numerical

Let's look at $s(x) = \sin(x)$.

If we build a table of values extending towards infinity...

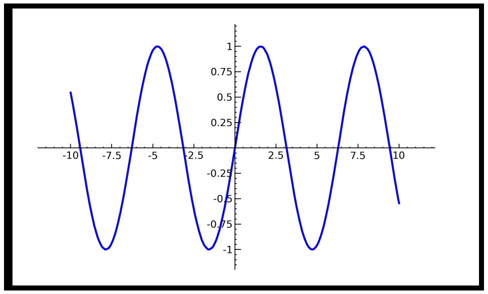
x	1.0	10.0	100.0	1000.0	10000.0	1,000,000,000.0
y	0.8415	-0.5540	-0.5063	0.8269	-0.3056	0.5458

...we don't see any trend to *one* special number.

Second example: graphical

Let's look at $s(x) = \sin(x)$.

If we plot the function for intervals with larger and larger x values...



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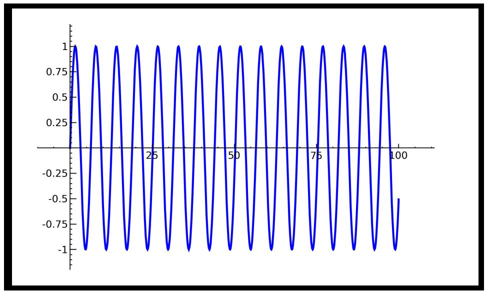
Let's look at $s(x) = \sin(x)$.

If we plot the function for intervals with larger and larger x values…



…we see that s oscillates between -1 and 1. Still there is no trend to *one* special value.

It looks as if there is no limit.

Second example: precise

Let's look at $s(x) = \sin(x)$.

Recall the precise definition.

Definition: *(Limit at ∞)*
$\lim_{x\to\infty} f(x) = L$ if

- for all $\varepsilon > 0$,
- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

In our case,

- For any hypothetical limit L,
- if $\varepsilon = 0.9$,
- then regardless of the value of N,
- we can find infinitely many x values
- such that $|\sin(x) - L| > \varepsilon$.

Second example: precise

Let's look at $s(x) = \sin(x)$.

Recall the precise definition.

Definition: *(Limit at ∞)*
$\lim_{x\to\infty} f(x) = L$ if

- for all $\varepsilon > 0$,
- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

For any hypothetical limit L, if $\varepsilon = 0.9$, then regardless of the value of N, we can find infinitely many x values such that $|\sin(x) - L| > \varepsilon$.

For example, if $L = 0$, then $s\left(\frac{\pi}{2}\right) = 1$, $s\left(\frac{5\pi}{2}\right) = 1$, $s\left(\frac{9\pi}{2}\right) = 1$, $\ldots$. This pattern continues for ever, so the limit of the y values does not approach 0 even though $y = 0$ infinitely many times at $x = 0, \pi, 2\pi, \ldots$.

Second example: precise

Let's look at $s(x) = \sin(x)$.

Recall the precise definition.

Definition: *(Limit at ∞)*
$\lim_{x\to\infty} f(x) = L$ if

- for all $\varepsilon > 0$,
- there exists N such that for *every* $x > N$
- $|f(x) - L| < \varepsilon$.

For any hypothetical limit L, if $\varepsilon = 0.9$, then regardless of the value of N, we can find infinitely many x values such that $|\sin(x) - L| > \varepsilon$.

We could play this same game with *any* value of L. (Try it!) So,

$$\lim_{x\to\infty} s(x) \text{ does not exist.}$$

Third example

Finally, we'll look at $d(x)=1/x$.

Third example: note

Finally, we'll look at $d(x)=1/x$.

This is an important and useful function. Along with its limits, $d(x)$ recurs quite a bit in Calculus.

Third example: numerical approach

Finally, we'll look at $d(x)=1/x$.

Use the numerical approach to fill in the question marks, and obtain an estimate for $\lim_{x\to\infty} 1/x$.

x	1.0	10.0	100.0	1000.0	10000.0	1,000,000,000.0
y	?	?	?	?	?	?

The values *should* seem to approach a number.

Third example: numerical approach

Finally, we'll look at $d(x)=1/x$.

Use the numerical approach to fill in the question marks, and obtain an estimate for $\lim_{x\to\infty} 1/x$.

x	1.0	10.0	100.0	1000.0	10000.0	1,000,000,000.0
y	1.0	0.1	0.01	0.001	0.0001	0.000000001

The values *should* seem to approach a number.

Third example: graphical approach

Finally, we'll look at $d(x) = 1/x$.

Use the numerical approach to fill in the question marks, and obtain an estimate for $\lim_{x\to\infty} 1/x$.

x	1.0	10.0	100.0	1000.0	10000.0	1,000,000,000.0
y	1.0	0.1	0.01	0.001	0.0001	0.000000001

Use the graphical approach to generate two or three plots, and check that the guess you'd obtain from the graph agrees with the guess that you'd obtain from your table.

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Third example: graphical approach

Finally, we'll look at $d(x) = 1/x$.

Use the numerical approach to fill in the question marks, and obtain an estimate for $\lim_{x\to\infty} 1/x$.

x	1.0	10.0	100.0	1000.0	10000.0	1,000,000,000.0
y	1.0	0.1	0.01	0.001	0.0001	0.000000001

Use the graphical approach to generate two or three plots, and check that the guess you'd obtain from the graph agrees with the guess that you'd obtain from your table.

You should have obtained the estimate

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Let's check this using the precise definition!

Third example: precise

For any ε, we want to find a value of N such that if $x > N$ then $|1/x - 0| < \varepsilon$. That is,

$$\left|\frac{1}{x}\right| < \varepsilon.$$

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Third example: precise

For any ε, we want to find a value of N such that if $x > N$ then $|1/x - 0| < \varepsilon$. That is,

$$\left|\frac{1}{x}\right| < \varepsilon$$

$$-\varepsilon < \frac{1}{x} < \varepsilon.$$

(Property of this absolute value inequality.)

Third example: precise

For any ε, we want to find a value of N such that if $x > N$ then $|1/x - 0| < \varepsilon$. That is,

$$-\varepsilon < \frac{1}{x} < \varepsilon$$
$$-\varepsilon x < 1 < \varepsilon x.$$

(Multiplied both sides by x.
Since $x > 0$ as $x \to \infty$,
we don't have to change the direction of the inequality.)

Third example: precise

For any ε, we want to find a value of N such that if $x > N$ then $|1/x - 0| < \varepsilon$. That is,

$$-\varepsilon x < 1 < \varepsilon x$$
$$-\varepsilon x < 1 \text{ and } 1 < \varepsilon x.$$

(Meaning of this inequality.)

Third example: precise

For any ε, we want to find a value of N such that if $x > N$ then $|1/x - 0| < \varepsilon$. That is,

$$-\varepsilon x < 1 \text{ and } 1 < \varepsilon x$$
$$\text{so } x > -\frac{1}{\varepsilon} \text{ and } x > \frac{1}{\varepsilon}.$$

Third example: precise

For any ε, we want to find a value of N such that if $x > N$ then $|1/x - 0| < \varepsilon$. That is,

$$-\varepsilon x < 1 < \varepsilon x$$
$$\text{so } x > -\frac{1}{\varepsilon} \text{ and } x > \frac{1}{\varepsilon}.$$

The larger of these two values is $1/\varepsilon$.

If, for example, $\varepsilon = 0.1$, then every value of x that is *greater* than $1/\varepsilon = 10$ would satisfy $|d(x)| < \varepsilon$. (*Check* it!) This proves that $\lim_{x\to\infty} d(x) = 0$.

BIG-TIME FACT

As mentioned, the fact we just proved is something you should remember for future reference:

Theorem:

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

The other direction

In all the examples we looked at in this module, we focused on *positive* infinity. Similar arguments can be made for limits at *negative* infinity. The definition that you would use is

Definition: *(Limit at $-\infty$)*
$\lim_{x\to-\infty} f(x) = L$ if

- for all $\varepsilon > 0$,
- there exists N such that for *every* $x < N$
- $|f(x) - L| < \varepsilon$.

In this case, multiplying or dividing the inequalities by x would require you to change the direction of the inequality.

Conclusion

- When we want to analyze how the y values of a function f behave when x increases (or decreases) without bound, we talk about this as *the limit of f as x approaches infinity (or negative infinity).*
- This limit is defined as

  Definition: *(Limit at ∞)*
  $\lim_{x\to\infty} f(x) = L$ if

  - for all $\varepsilon > 0$,
  - there exists N such that for *every* $x > N$
  - $|f(x) - L| < \varepsilon$.
- As with limits at finite values, there can only be one value for such a limit, and they do not always exist.
- All three methods of estimating or evaluating a limits—numerical, graphical, and precise—can be used for limits at infinity.
- An important limit at infinity to remember is $\lim_{x\to\infty} 1/x = 0$.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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