Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

The derivative of the exponential function

In this module we determine more efficient ways of computing the derivatives of *exponential functions*. (Recall that an *exponential function* has the form a^x where the base a is a positive real number, not equal to 1. An important exponential function uses the base $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.7183$.) As with polynomials, we use the precise definition of the derivative.

Defining the problem

Question: How do we evaluate the derivative of an exponential function $f(x) = a^x$?

SAGE worksheets

You will need a blank SAGE worksheet for this module.

A difficulty with the precise definition

The precise definition of the derivative tells us that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

For an exponential function $f(x) = a^x$, this evaluates to

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}.$$

How do we simplify this expression? We would like a shortcut, so as to avoid it.

The derivative of e^x

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Rewrite the difference quotient found on page **??**:

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$$f'(x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$
$$=\lim_{h\to 0}\frac{e^x\cdot e^h-e^x}{h}.$$

(A property of exponents.)

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Rewrite the difference quotient found on page ??:

$$f'(x)=\lim_{h\to 0}\frac{e^x\cdot e^h-e^x}{h}$$
$$=\lim_{h\to 0}\frac{e^x(e^h-1)}{h}.$$

(Factored a common e^x.)

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Rewrite the difference quotient found on page ??:

$$f'(x)=\lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$
$$=e^x\lim_{h\to 0}\frac{e^h-1}{h}.$$

(Since e^x is constant *with respect to* h,
we can factor it from the limit.)

The derivative of e^x

So far we have reasoned that if $f(x)=e^x$, then

$$f'(x)=e^x\lim_{h\to 0}\frac{e^h-1}{h}.$$

If we could determine the value of the limit, we would be done!

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We cannot substitute $h = 0$ into the expression, because the function is not continuous at $h = 0$ (division by zero). However, you have learned how to use SAGE to estimate the value of the limit. Use it now to estimate, *both numerically and graphically*, the value of the limit.

Module: The derivative of the exponential function 5

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Hint 1: To evaluate it numerically, define the function $f = (e^h - 1)/h$ and substitute values of h that approach zero. You can use the commands

```
var('h')
f = (e^h-1)/h
[round(f(x),4) for x in [1,0.5,0.1,0.01,0.001,-0.001,-0.01,-0.1,-0.5,-1]]
```

Module: The derivative of the exponential function 5

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You should find

h	1	0.5	0.1	0.01	0.001	0	-0.001	-0.01	-0.1	-0.5	-1
$f(h)$	1.7183	1.2974	1.0517	1.005	1.0005	??	0.9995	0.995	0.9516	0.7869	0.6321

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$f(h)$	1.7183	1.2974	1.0517	1.005	1.0005	1	0.9995	0.995	0.9516	0.7869	0.6321

It *looks* as if $f(h) \to 1$ when $h \to 0$. See if a graph gives the same intuition.

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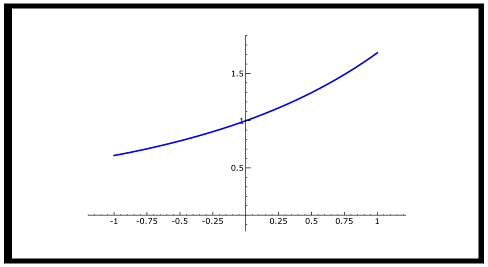
If we could determine the value of the limit, we would be done!

You should see a graph that looks like this:



The graph appears to confirm our belief that the limit is 1.

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$$f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}.$$

If we could determine the value of the limit, we would be done!

We will assume that our intuition is correct. Thus

$$f'(x) = e^x \cdot 1 = e^x.$$

In other words, e^x *is its own derivative*!

BIG-TIME FACT

Theorem: *(e^x is its own derivative)*
If $f(x) = e^x$, then $f'(x) = e^x$. Written another way,

$$y = e^x \qquad \Longrightarrow \qquad y' = y.$$

(We have argued intuitively that this is true, but our argument does not explain why

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

It can be proved using the definition

$$e^h = \lim_{x \to \infty} \left(1 + \frac{h}{x}\right)^x,$$

using an argument similar to (but harder than) the one for the derivative of a monomial.)

Example

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$$\frac{d}{dx} e^{4x-3} = \frac{dg}{du} \cdot \frac{du}{dx}.$$

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(Substituted for the derivatives.)

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Here the "inside" function is $u(x)=4x-3$, and the "outside" function is $g(u)=e^u$. From the Chain Rule,

$$\begin{aligned}\frac{d}{dx}e^{4x-3} &= e^u \cdot 4\\ &= 4e^{4x-3}.\end{aligned}$$

(Substituted for u.)

The derivative of a^x, for arbitrary a

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In that case, we can say that

$$a^x = e^{\ln(a^x)}$$

(Inverse functions.)

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(Property of logarithms.)

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$$\begin{aligned} a^x &= e^{x \ln a} \\ &= (e^x)^{\ln a} \end{aligned}$$

(Property of exponents.)

The derivative of a^x, for arbitrary a

To determine the derivative of a^x for other bases, we use an algebraic technique along with *the Chain Rule*.

From the algebraic technique, we have $a^x = (e^x)^{\ln a}$. Now we'll apply the Chain Rule. Here the "inside" is $u(x) = e^x$ and the "outside" is $g(u) = u^{\ln a}$.

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Thus

$$\frac{d}{dx}(e^x)^{\ln a} = \ln a \cdot (e^x)^{\ln a - 1} \cdot (e^x).$$

(The Chain Rule, and substitution.)

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Thus

$$\begin{aligned}\frac{d}{dx}(e^x)^{\ln a} &= \ln a \cdot (e^x)^{\ln a - 1} \cdot (e^x) \\ &= \ln a \cdot (e^x)^{\ln a \cancel{-1+1}^{0}}.\end{aligned}$$

(A property of exponents *that have the same base*. In this case the base is e^x.)

Module: The derivative of the exponential function 9

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Thus

$$\begin{aligned}\frac{d}{dx}(e^x)^{\ln a} &= \ln a \cdot e^{\ln a^x} \\ &= \ln a \cdot a^x.\end{aligned}$$

(e^x and $\ln x$ are inverse functions.)

The derivatives of exponential functions

We have finally found the derivatives we wanted!

Theorem:

- The derivative of e^x is itself.
- The derivative of a^x is $a^x \ln a$.

You should commit these to memory.

Examples

As with other derivative "shortcuts", we can use them with the properties of the derivative. For example,

$$\frac{d}{dx}(2e^x) = 2\frac{d}{dx}e^x = 2x$$
$$\frac{d}{dx}(xe^x) = e^x + xe^x$$
$$\frac{d}{dx}\left(\frac{2^x}{x^2}\right) = \frac{2^x(\ln 2)x^2 - 2^{x+1}x}{x^4}$$
$$\frac{d}{dx}\left(e^{x^2}\right) = 2x \cdot e^{x^2}$$

These illustrate the constant multiple rule, the product rule, the quotient rule, and the Chain Rule, respectively.

Conclusion

- We found a shortcut for the derivative of e^x. Interestingly, e^x is its own derivative.

  Theorem: If $f(x) = e^x$, then $f'(x) = e^x$. Written another way,
  $$y = e^x \qquad \Longrightarrow \qquad y' = y.$$

- To find this shortcut, we used the precise definition of the derivative and an intuitive estimate of the limit
  $$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$
- We used the shortcut to find a shortcut for the derivative of a^x where the base a is any positive real number, not equal to one.

  Theorem: The derivative of a^x is $a^x \ln a$.

- To find this second shortcut, we made use of properties of exponents and logarithms, along with the Chain Rule.
- These shortcuts can be combined with the properties of derivatives.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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