Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Linear Approximations

In this module we explore how to approximate a value of a function using the line tangent to the function at a nearby point. This method is known as linear approximation. We will then explore a related idea known as relative rates of change.

Sample problems

Sample problems:

- Approximate the value of $\sqrt[5]{2}$ to the tenths place.
- Suppose that blood flows along a blood vessel. The volume of blood per unit time that flows past a given point is called the *flux* F. Poiseulle's Law says that this value is proportional to the fourth power of the radius R of the blood vessel; that is,

$$R = kF^4.$$

  What is ratio of the relative change in F to the relative change in R, and how will a 5% increase in the radius affect the flow of blood?

SAGE worksheets

For this module you will not need a SAGE worksheet, although you can perform some of the calculations and plots yourself.

Linear Approximations: the idea

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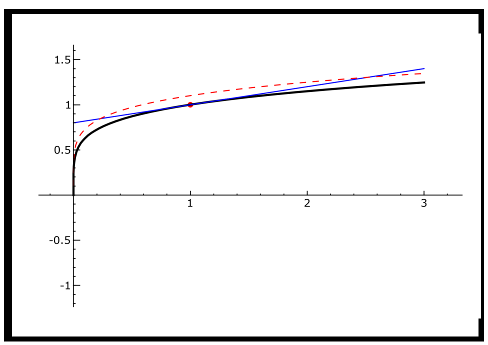
For example, if $f(x) = \sqrt[5]{x}$, the graph below shows both f and the line L tangent to f at $x = 1$.



The dashed red line indicates a boundary of $1/10$ from f. It shows that $L(x) - f(x) < 1/10$ on the interval $[0.25, 2.5]$. Thus, if you want to compute $\sqrt[5]{2}$ and you don't mind an error of less than $1/10$, you could compute $L(2)$ instead.

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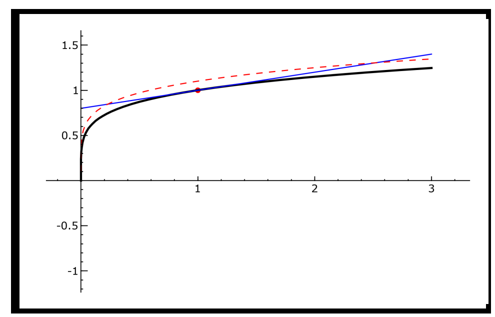


On the other hand, if you travel too far from $x = 1$, the error is too large. If you wanted to compute $\sqrt[5]{3}$, the dashed red line lies between the tangent line and the curve $f(x)$. The linear approximation would give an error greater than $1/10$.

Linear Approximations

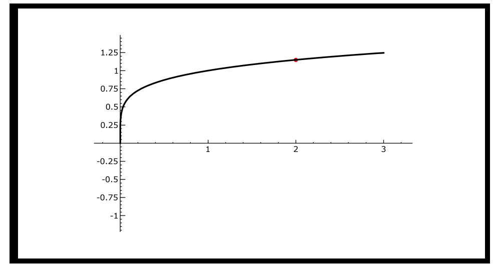
In general, the strategy to compute a linear approximation of $f(a)$ is as follows:

- Identify the function $f(x)$.
- Identify a point b such that
  - b is close to a, and
  - both $f(b)$ and $f'(b)$ are *easy to compute*.
- Determine $L(x)$, the equation of the line tangent to f at $x = b$.
- Compute $L(b)$; that is, substitute $x = b$ into $L(x)$.

$L(b)$ is the linear approximation of $f(a)$.

Linear Approximations

We can illustrate what we are doing with several graphs. We return to approximating $\sqrt[5]{2}$.



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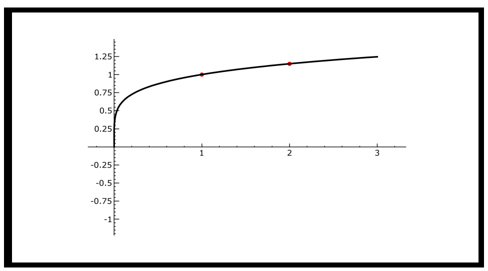

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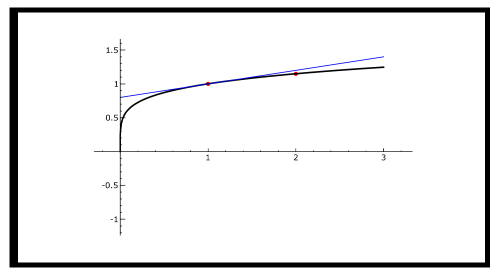

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- For a "nice" point that is easy to evaluate and close to $a = 2$ we use $b = 1$.
- The line tangent to $f(x)$ at $x = b$ is $L(x) = \frac{1}{5}(x-1)+1$.
  (You should check this to make sure you can obtain it yourself.)

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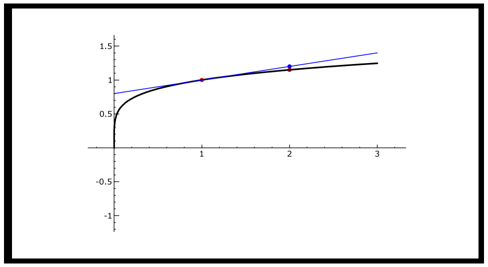

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- We evaluate $L(a)$, which is $L(2)$, and we get the point shown on the line.

The estimate comes to $L(2)=\frac{1}{5}(2-1)+1=1.2$.

Module: Linear Approximation 5

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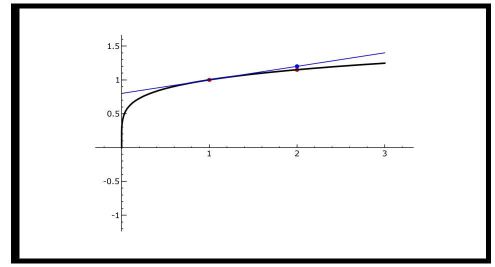

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- We evaluate $L(a)$, which is $L(2)$, and we get the point shown on the line.

The estimate comes to $L(2)=\frac{1}{5}(2-1)+1=1.2$. We conclude that $\sqrt[5]{2}\approx 1.2$. This is within one-tenth of the true value.

Applied Problem

Earlier we posed the problem about blood flow:

> **Question:** Suppose that blood flows along a blood vessel. The volume of blood per unit time that flows past a given point is called the *flux* F. Poiseulle's Law says that this value is proportional to the fourth power of the radius R of the blood vessel; that is,
>
> $$R=kF^4.$$
>
> What is the ratio of the relative change in F to the relative change in R, and how will a 5% increase in the radius affect the flow of blood?

Solving this problem uses a similar idea.

Differentials

Recall the *Leibniz notation* of the derivative. For any point x where f is differentiable,

$$f'(x)=\frac{dy}{dx}.$$

Remember also that dy/dx was not a fraction, because dx and dy had no individual meaning. Now we give them a meaning.

> **Definition:** We say that the quantities dx and dy are *differentials* when the satisfy the property
>
> $$dy=f'(x)\,dx.$$
>
> Here dy is a dependent variable, whose quantity depends on the values of x and dx.

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In a linear approximation, we also call relative change the *relative error*.

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From the definition of a differential, we know that

$$dF = F'\, dR$$
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Module: Linear Approximation 9

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By substitution,

$$\frac{\text{relative change in } F}{\text{relative change in } R} = \frac{\frac{dF}{F}}{\frac{dR}{R}} = \frac{\frac{4kR^3\, dR}{kR^4}}{\frac{dR}{R}} = \frac{4\, dR}{R} \cdot \frac{R}{dR} = 4.$$

The ratio of the relative rate of change of F to the relative rate of change of R is 4.

Applied Problem: second part

The second part of the problem asked,

Question: How will a 5% increase in the radius affect the flow of blood?

Module: Linear Approximation 10

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It asks for the relative rate change of change in F. In the first part of the problem, we learned that the ratio between the two is

$$\frac{dF/F}{dR/R} = 4.$$

The relative change of F is 4 times the relative change of R. Thus, the flow of blood should increase by 20%.

Conclusion

In this module, you have seen:

- how to compute a linear approximation of an irrational number by using the line tangent to a function;
    - Identify the function $f(x)$.
    - Identify a point b such that
        - b is close to a, and
        - both $f(b)$ and $f'(b)$ are *easy to compute*.
    - Determine $L(x)$, the equation of the line tangent to f at $x = b$.
    - Compute $L(b)$; that is, substitute $x = b$ into $L(x)$.
- how a linear approximation can lose its accuracy as one moves away from the point where the line is built; and
- how differentials are used to compute relative rates of change.

Later you will learn a method of approximating irrational numbers that is usually a better choice, called *Newton’s Method*.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

Department of Mathematics at
The University of
Southern Mississippi