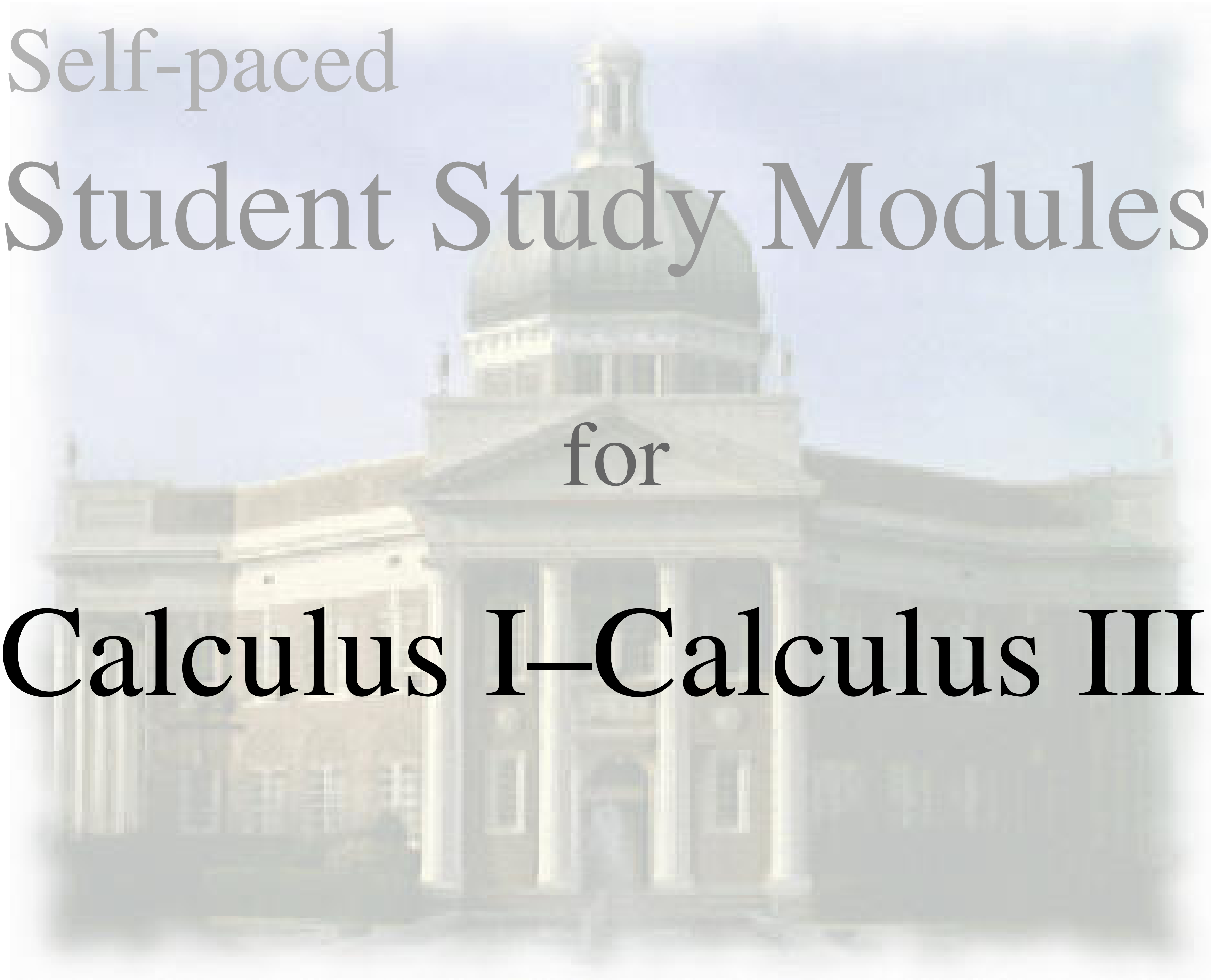


Self-paced

# Student Study Modules

for

# Calculus I–Calculus III

# Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages **⟨Next⟩**, backward **⟨Prev⟩**, or view all the slides in this tutorial **⟨Index⟩**.
- The **⟨Back to Calc I⟩** button returns you to the course home page.
- A full symbolic algebra package **⟨Sage⟩** is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text **⟨CalcText⟩** provides a quick search of basic calculus topics.
- You can get help from Google Calculus **⟨GoogleCalc⟩**.
- A monochrome copy of this module is suitable for printing **⟨Print⟩**.

When all else fails, feel free to contact your instructor.

# Overview

In the previous module, you studied the *intuitive* idea of a derivative. In this module, you will learn the *precise* definition of the derivative. You will also see how the precise definition is related to limits.

# A Sample problem

A supervillain throws a superhero off the roof of the Empire State Building in New York. This particular superhero cannot fly, but he can bounce off the ground. Having nothing in particular to think about as he falls, he wonders how fast he travels at any given moment.

He knows from physics that an object dropped above the earth obeys the function

$$H(t) = -9.8t^2 + h_0$$

where

- $H(t)$ is the height of the object above the ground,
- $t$ is the time in seconds, and
- $h_0$ is the height of the object at time $t = 0$.

The roof of the Empire State Building is 1250 ft above the ground.

**Question:** How fast is the superhero falling after five seconds?

# SAGE worksheets

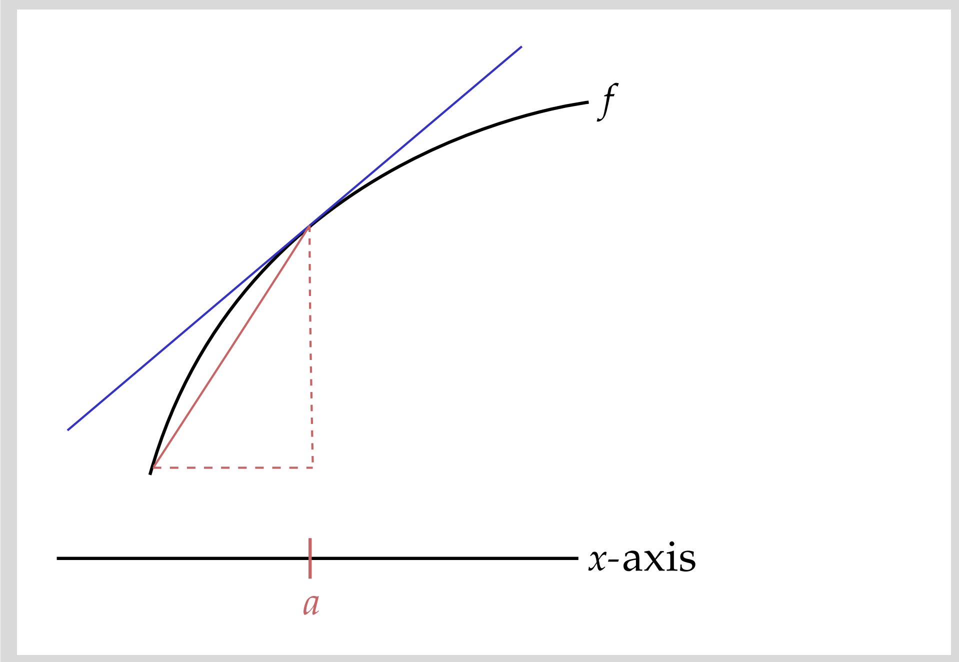
In this module you will not need a SAGE worksheet.

# Precise definition of the derivative

Recall from the previous module that the derivative is *the instantaneous rate of change*. We had described this as *the slope of the line tangent to* the curve. We further explained that this slope was *the limit of the slopes of the secant lines*.

# Precise definition of the derivative

Let $f(x)$ be a function, and $a$ a real number. To find the derivative of $f$ at $x=a$, we need to find the limit of the slopes of the secant lines passing through $x=a$.



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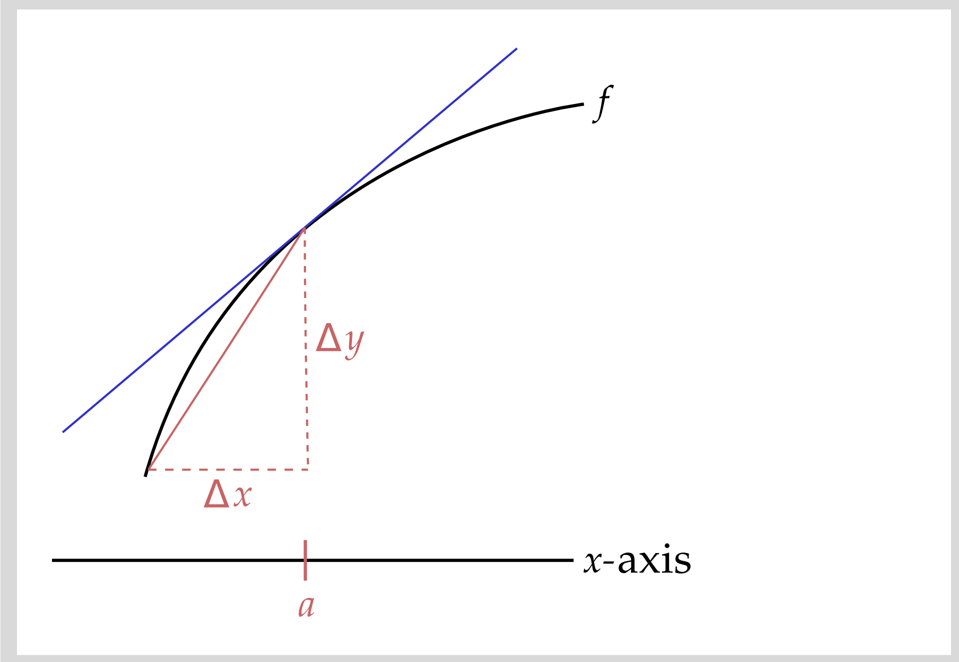


That means the derivative of $f(x)$ is

$$\lim_{x \to a} \frac{\Delta y}{\Delta x}.$$

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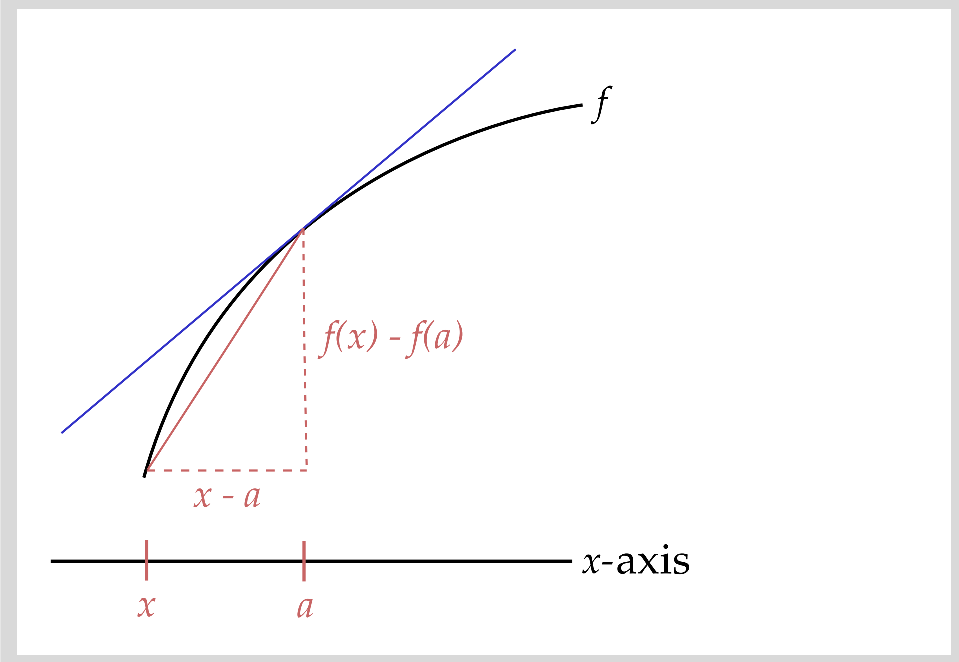


If we let $x$ vary around $a$, then $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$. That means the derivative of $f(x)$ is

$$\lim_{x \to a} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

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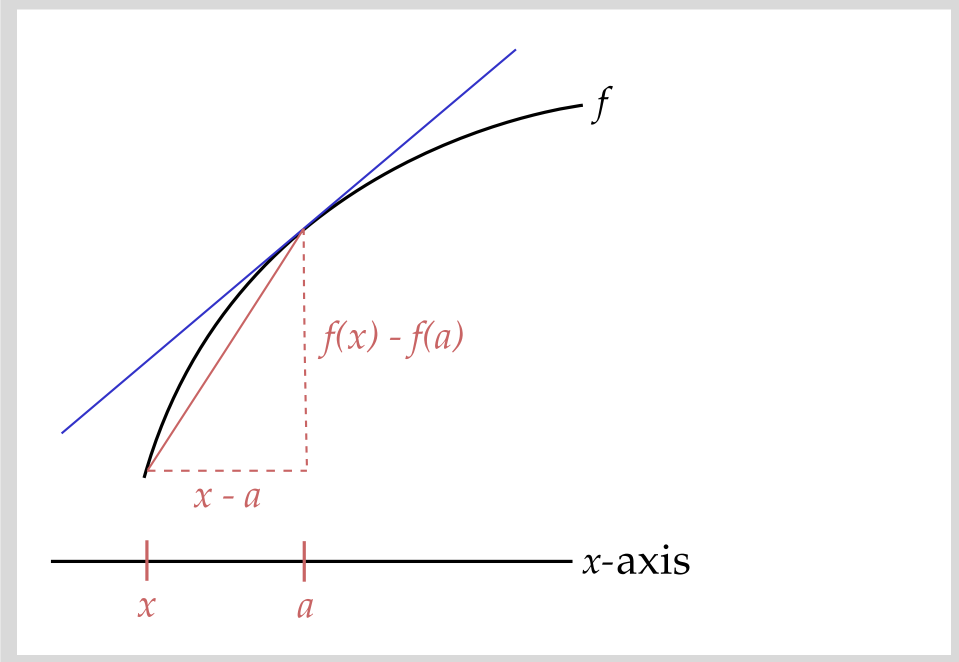
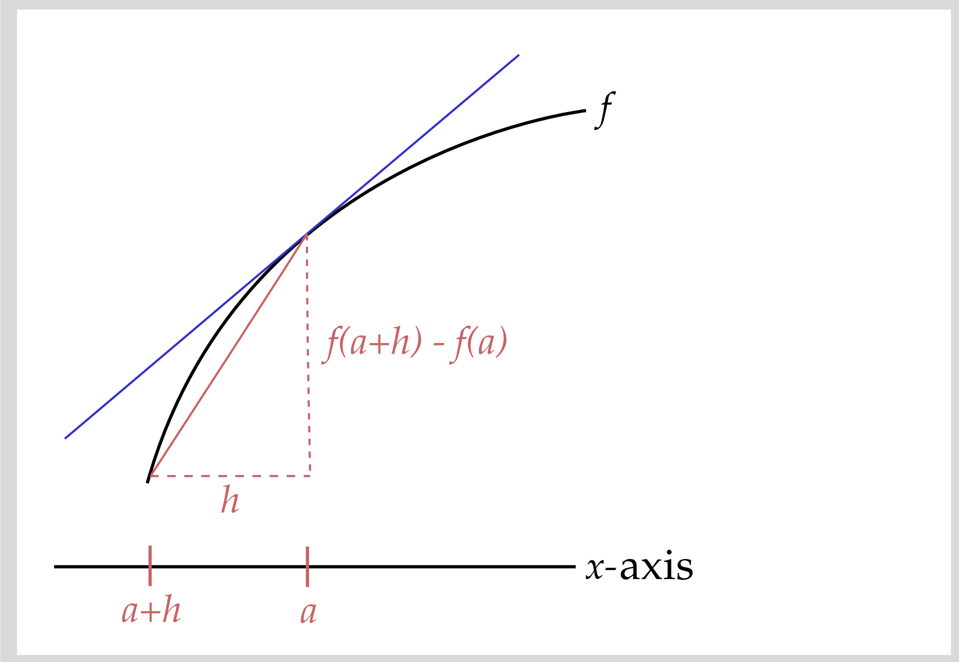



We usually write $f'(a)$ for the derivative of $f(x)$ at $x=a$, so

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{\Delta y}{\Delta x} \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}. \end{aligned}$$

# Alternate definition

Let $f(x)$ be a function, and $a$ a real number. To find the derivative of $f$ at $x=a$, we need to find the limit of the slopes of the secant lines passing through $x=a$.




A second definition considers $h=x-a$. In this case the definition becomes

$$f'(a)=\lim_{x\to a}\frac{\Delta y}{\Delta x}$$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

# Two definitions

So there are two “precise” definitions of the derivative that you must keep in mind.

**Definition:** *(The derivative at a point)*

The derivative of a function $f$ at $x = a$ is

- $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, or equivalently
- $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

You should think about why these two definitions are equivalent both algebraically and geometrically. Re-examine the previous slides and compare the diagrams until you understand how they are the same.

# Two notations for the derivative

We usually write $y'$ for the derivative of $y$; this is called Newton's notation. We can also write $y''$ for the derivative of the derivative, called the second derivative; $y'''$ for the derivative of the second derivative, called the third derivative; and so forth.

Another notation is Leibniz notation:

$$y' \to \frac{dy}{dx}$$
$$y'' \to \frac{d^2y}{dx}$$
$$y''' \to \frac{d^3y}{dx}$$
$$\vdots$$

This is related to the fact that

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$y'' = \lim_{\Delta x \to 0} \frac{\Delta y'}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta \left( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \right)}{\Delta x}$$
$$\vdots$$

# Two notations for the derivative

We usually write $y'$ for the derivative of $y$; this is called Newton's notation. Likewise one can write $y''$ for the derivative of the derivative, called the second derivative; $y'''$ for the derivative of the second derivative, called the third derivative; and so forth.

Another notation you may see is Leibniz notation:

$$\begin{aligned} y' &\to \frac{dy}{dx} \\ y'' &\to \frac{d^2y}{dx} \\ y''' &\to \frac{d^3y}{dx} \\ &\vdots \end{aligned}$$

Do not confuse the notation $dy/dx$ for a fraction of two quantities $dy$ and $dx$; it is simply a way of writing the derivative.

# An example

We illustrate both definitions with the example given at the beginning of the module. We had $H(t) = -9.8t^2 + h_0$.

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We want to know

**Question:** How fast is the superhero falling after five seconds?

"How fast" is a *rate of change* of position. (In physics, we call this the *velocity*.) So we need to compute the derivative of $H(t)$ at $t = 5$.

That is, we need to compute $H'(5)$.

# $H'(5)$: method 1

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 1,

$$H'(5) = \lim_{t \to 5} \frac{H(t) - H(5)}{t - 5}.$$

We can't substitute $t = 5$, since that would give us zero in the denominator. We have to invest some effort into rewriting the fraction.

# $H'(5)$: method 1

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 1,

$$H'(5) = \lim_{t \to 5} \frac{H(t) - H(5)}{t - 5}$$

$$H'(5) = \lim_{t \to 5} \frac{(-9.8t^2 + \cancel{1250}) - (-9.8(5)^2 + \cancel{1250})}{t - 5}.$$

Notice that 1250 cancels.

# $H'(5)$: method 1

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$$H'(5) = \lim_{t \to 5} \frac{(-9.8t^2 + \cancel{1250}) - (-9.8(5)^2 + \cancel{1250})}{t-5}$$

$$H'(5) = \lim_{t \to 5} \frac{-9.8t^2 + 9.8(5)^2}{t-5}.$$

The two terms of the numerator have a common factor, 9.8. Let's rewrite by factoring it out.

# $H'(5)$: method 1

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 1,

$$H'(5) = \lim_{t \to 5} \frac{-9.8t^2 + 9.8(5)^2}{t-5}$$

$$H'(5) = \lim_{t \to 5} \frac{-9.8(t^2 - 25)}{t-5}.$$

Now recall from algebra that $t^2 - 25 = (t-5)(t+5)$. Let's rewrite it that way…

# $H'(5)$: method 1

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 1,

$$H'(5) = \lim_{t\to 5} \frac{-9.8(t^2-25)}{t-5}$$

$$H'(5) = \lim_{t\to 5} \frac{(-9.8(t+5)\cancel{(t-5)}}{\cancel{t-5}}.$$

…allowing us to divide out the $t-5$'s that appear in the numerator and the denominator.

# $H'(5)$: method 1

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 1,

$$H'(5) = \lim_{t \to 5} \frac{(-9.8(t+5)\cancel{(t-5)})}{\cancel{t-5}}$$

$$H'(5) = \lim_{t \to 5} -9.8(t+5).$$

We no longer have division by zero, so we can substitute $t = 5$ into the expression, and…

# $H'(5)$: method 1

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 1,

$$H'(5) = \lim_{t \to 5} -9.8(t+5)$$
$$H'(5) = -98.$$

...the superhero is falling at 98 ft/sec. (The negative indicates that he is falling.)

## $H'(5)$: method 1

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 1,

$$H'(5) = \lim_{t \to 5} -9.8(t+5)$$
$$H'(5) = -98.$$

…the superhero is falling at 98 ft/sec. (The negative indicates that he is falling.)

We could also write that $\frac{dH}{dt} = -98$ (Leibniz notation).

## $H'(5)$: method 2

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 2,

$$H'(5) = \lim_{h \to 0} \frac{H(5+h) - H(5)}{h}.$$

Again, we cannot substitute $h = 0$ to evaluate the limit, so we have to rewrite the fraction. We start by evaluating $H(5+h)$ and $H(5)$.

# $H'(5)$: method 2

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 2,

$$H'(5) = \lim_{h\to 0} \frac{H(5+h)-H(5)}{h}$$

$$H'(5) = \lim_{h\to 0} \frac{(-9.8(5+h)^2+\cancel{1250})-(-9.8\cdot 5^2+\cancel{1250})}{h}.$$

Notice that 1250 cancels again this time. We'll simplify the numerator next.

# $H'(5)$: method 2

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 2,

$$H'(5) = \lim_{h \to 0} \frac{(-9.8(5+h)^2 + \cancel{1250}) - (-9.8 \cdot 5^2 + \cancel{1250})}{h}$$

$$H'(t) = \lim_{h \to 0} \frac{-9.8(25 + 10h + h^2) + 9.8 \cdot 25}{h}.$$

We need to distribute $-9.8$.

# $H'(5)$: method 2

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 2,

$$H'(t) = \lim_{h \to 0} \frac{-9.8(25 + 10h + h^2) + 9.8 \cdot 25}{h}$$

$$H'(t) = \lim_{h \to 0} \frac{\cancel{-9.8 \cdot 25} - 98h - 9.8h^2 + \cancel{9.8 \cdot 25}}{h}.$$

Again we have cancellation. When using this second definition of the derivative, *it is common for terms that do not contain $h$ to cancel.*

# $H'(5)$: method 2

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 2,

$$H'(t) = \lim_{h\to 0} \frac{\cancel{-9.8\cdot 25} - 98h - 9.8h^2 + \cancel{9.8\cdot 25}}{h}$$

$$H'(t) = \lim_{h\to 0} \frac{-98h - 9.8h^2}{h}.$$

Notice that $h$ is a factor of every term in the numerator. Let's factor it out.

# $H'(5)$: method 2

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 2,

$$H'(t) = \lim_{h \to 0} \frac{-98h - 9.8h^2}{h}$$

$$H'(t) = \lim_{h \to 0} \frac{\cancel{h}(-98 - 9.8h)}{\cancel{h}}.$$

This allows us to divide out $h$, eliminating it from the denominator.

# $H'(5)$: method 2

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 2,

$$H'(t) = \lim_{h\to 0} \frac{\cancel{h}(-98 - 9.8h)}{\cancel{h}}$$

$$H'(5) = \lim_{h\to 0}(-98 - \cancelto{0}{9.8h}).$$

We can now substitute $h = 0$…

# $H'(5)$: method 2

Since we gave you two definitions of the derivative, we illustrate this computation two different ways. Using definition 2,

$$H'(5) = \lim_{h\to 0}(-98 - \cancel{9.8h}^{\,0})$$

$$H'(5) = -98.$$

...we obtain the same answer as before: the superhero is falling at 98 ft/sec.

# Algebra

You will have noticed that it was necessary to perform *some* sort of algebraic manipulation whenever applying the precise definition of the derivative. This is usually the case. Techniques we used in this example were:

- distribution,
- factoring, and
- dividing common factors in fractions.

# Algebra

You will have noticed that it was necessary to perform *some* sort of algebraic manipulation whenever applying the precise definition of the derivative. This is usually the case. Techniques we used in this example were:

- distribution,
- factoring, and
- dividing common factors in fractions.

You will also find it necessary to use other algebraic techniques, such as

- rationalizing the denominator (multiply by conjugate), and
- simplifying complex fractions (multiply numerator by reciprocal of denominator).

You will also need to use techniques for taking limits of trigonometric functions. This is covered in your text, and your instructor will also discuss it.

# Conclusion

You have now learned two different, but equivalent definitions of the derivative.

**Definition:** *(The derivative at a point)*
The derivative of a function $f$ at $x=a$ is

- $f'(a)=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$, or equivalently
- $f'(a)=\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

You should memorize them, but more importantly you should understand *why* the definition is defined with them. You will use them to compute many derivatives in the coming days.

# End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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