Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Overview

In this lesson you begin to study a topic called the *derivative*. In Calculus, the derivative is a tool used to describe the *rate of change* of a function.

Why would we want to compute the rate of change of a function? There are many examples, most of which you will see in your homework, of course. Generally speaking, the rate of change has a large number of important applications to real world problems because *real-world problems often involve changing quantities*.

A Sample problem

An electronic instrument makes a sound by vibrating a speaker's voice coil according to the function $s(x) = v\sin(x)$, where

- v is a constant;
- x is the time in seconds; and
- $s(x)$ is the distance in mm that the coil has moved from equilibrium.

Question: What is the rate of change of the position of the coil at time $x = 1$ second when $v = 1$ and when $v = 4$?

SAGE worksheets

In this lab you could use a blank SAGE worksheet, although you can get by with just a calculator, too.

Two kinds of rates of change

We said that the derivative is the rate of change. There are actually two different rates of change that you need to know, and only one of them is the derivative.

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The *average* rate of change between *two points* $x = a$ and $x = b$ is the slope of the line connecting the points $(a, f(a))$ and $(b, f(b))$.

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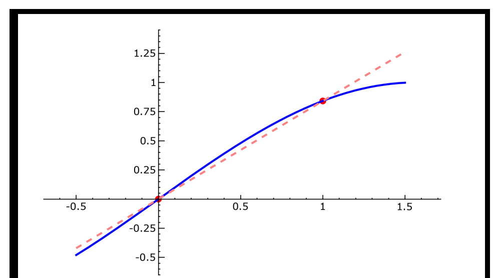
In the case of the speaker ($s(x) = \sin(x)$), the average rate of change between $x = 0$ and $x = 1$ is

$$\frac{\sin(1) - \sin(0)}{1 - 0} \approx 0.84.$$



Two kinds of rates of change

We said that the derivative is the rate of change. There are actually two different rates of change that you need to know, and only one of them is the derivative.

Definition: The *instantaneous* rate of change at *one point* $x = b$ is the slope of the line *tangent to* the curve at $x = b$.

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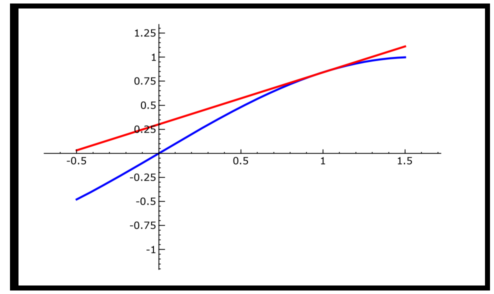
In the case of the speaker ($s(x) = \sin(x)$), the instantaneous rate of change at $x = 1$ is $m \approx 0.54$.



(If you have no idea how I computed 0.54, don't worry. You're not supposed to know that yet.)

Tangent = Limit of Secants

It might not be obvious at first, but as a approaches b, the secant lines between $x = a$ and $x = b$ usually approach the tangent line at $x = b$. The next few slides illustrate this.

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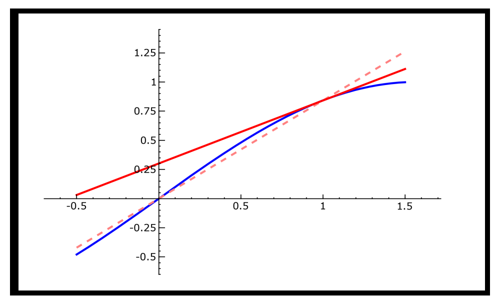

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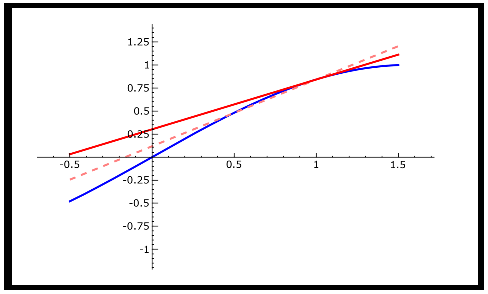
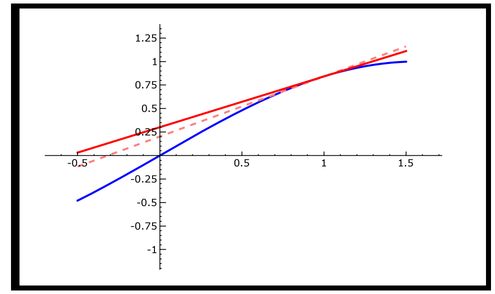
The secant line between $x = 0.5$ and $x = 1$:



Tangent = Limit of Secants

It might not be obvious at first, but as a approaches b, the secant lines between $x = a$ and $x = b$ usually approach the tangent line at $x = b$. The next few slides illustrate this.

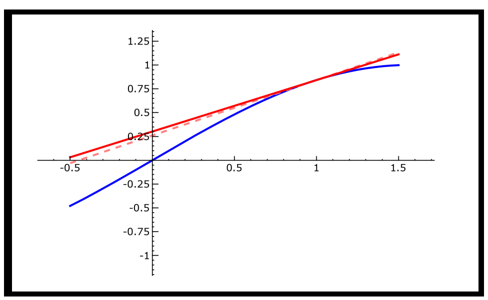
The secant line between $x = 0.75$ and $x = 1$:



Tangent = Limit of Secants

It might not be obvious at first, but as a approaches b, the secant lines between $x = a$ and $x = b$ usually approach the tangent line at $x = b$. The next few slides illustrate this.

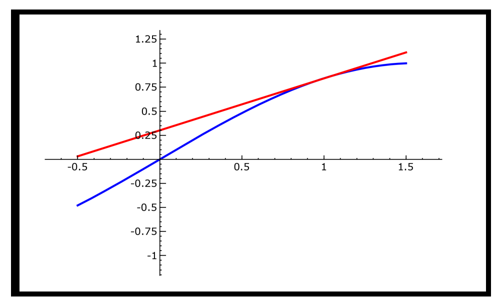
The secant line between $x = 0.9$ and $x = 1$:



Tangent = Limit of Secants

It might not be obvious at first, but as a approaches b, the secant lines between $x = a$ and $x = b$ usually approach the tangent line at $x = b$. The next few slides illustrate this.

The tangent line at $x = 1$:



Okay, so which one is the derivative?

Definition: *(The derivative at a point)*
The *derivative* of f at any point $x = b$ is the *instantaneous* rate of change of f.

As we pointed out earlier, this is the slope of the tangent line. Thus the derivative is the *slope* of the line tangent to f at $x = b$.

Module: an intuitive notion of derivatives 5

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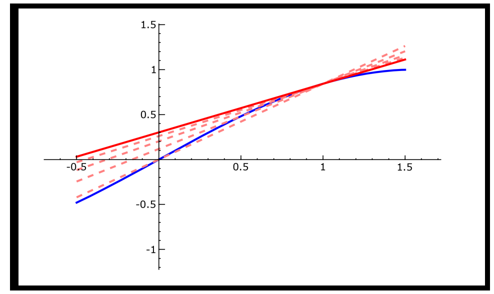
So the derivative is the *limit* of the slopes of the secant lines between $x = a$ and $x = b$ as $x = a$ approaches $x = b$.



Definition of the derivative

We write this formally as

Definition: *(The derivative at a point)*

$$f'(b) = \lim_{x \to b} \frac{f(x) - f(b)}{x - b}$$

where $f'(b)$ is the symbol used for the derivative of f at $x = b$.

This assumes that the above limit exists.

Definition: If the derivative of f exists and is finite at $x = b$, we say that f is *differentiable at* $x = b$. If the derivative does not exist, then f is not differentiable at $x = b$.

Differentiability

An important point to remember is that

Theorem: If a function f is differentiable at $x = b$, then f is continuous at $x = b$.

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> **Theorem:** If a function f is differentiable at $x=b$, then f is continuous at $x=b$.

Why? The definition of continuity requires that $f(b)$ exists, and $\lim_{x\to b} f(x)=f(b)$.

- If $f(b)$ does not exist, then we cannot compute the derivative , whose definition requires $f(b)$. (*Go back and check where!*)
- If $\lim_{x\to b} f(x)\neq f(b)$, then the difference quotient
$$\frac{f(x)-f(b)}{x-b}$$
approaches a non-zero constant in the numerator, and zero in the denominator. In a previous module, we stated that this was *not a finite quantity*. Differentiability requires a *finite* limit for the difference quotient. (*Go back and check where!*)

Numerical approach

We turn our attention to the question of computing derivatives. Because it can be hard to estimate slopes from looking at a graph, it is often easy to estimate the derivative using the *numerical approach* to computing a limit.

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Build a table of a-m values, where m is the slope of the secant line of $s(x)=\sin(x)$ between $x=a$ and $x=1$, where $a=0, 0.9, 0.99, 0.9999, 1.0001, 1.01, 1.1, 2$.

a	0	0.9	0.99	0.9999	1	1.0001	1.01	1.1	2
m	?	?	?	?	???	?	?	?	?

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a	0	0.9	0.99	0.9999	1	1.0001	1.01	1.1	2
m	?	?	?	?	???	?	?	?	?

Remember that you can make SAGE do this computation using a `for` loop if you like.

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Build a table of a-m values, where m is the slope of the secant line of $s(x) = \sin(x)$ between $x = a$ and $x = 1$, where $a = 0, 0.9, 0.99, 0.9999$.

a	0	0.9	0.99	0.9999	1	1.0001	1.01	1.1	2
m	0.84	0.58	0.54	0.54	???	0.54	0.54	0.5	0.07

Notice that the slopes do indeed approach 0.54, which is what we had predicted before.

Example Problem

Now that we've touched on the concepts, let's look at the example problem again. It was:

An electronic instrument makes a sound by vibrating a speaker's voice coil according to the function $s(x) = v\sin(x)$, where

- v is a constant;
- x is the time in seconds; and
- $s(x)$ is the distance in mm that the coil moves.

Question: What is the rate of change of the position of the coil at time $x = 1$ second when $v = 1$ and when $v = 4$?

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Question: What is the rate of change of the position of the coil at time $x = 1$ second when $v = 1$ and when $v = 4$?

If you look back, you will see that we have in fact solved the problem for $v = 1$. We found that the derivative was $s'(1) \approx 0.54$.

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Remember that the derivative is the limit of the slopes of the secant lines. These slopes represent the ratio *change in s to change in x*. The problem told us that

- s represents distance in mm, and
- x represents time in secondes.

Thus the units should be mm/sec.

Example Problem: $v = 4$

Now that we've touched on the concepts, let's look at the example problem again. It was:

An electronic instrument makes a sound by vibrating a speaker's voice coil according to the function $s(x) = v\sin(x)$, where

- v is a constant;
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Question: What is the rate of change of the position of the coil at time $x = 1$ second when $v = 1$ and when $v = 4$?

Since we solved the problem for $v = 1$, take a few moments on your own to solve the problem for $v = 4$.

Example Problem: $v = 4$

Now that we've touched on the concepts, let's look at the example problem again. It was:

An electronic instrument makes a sound by vibrating a speaker's voice coil according to the function $s(x) = v\sin(x)$, where

- v is a constant;
- x is the time in seconds; and
- $s(x)$ is the distance in mm that the coil moves.

What is the rate of change of the position of the coil at time $x = 1$ second when $v = 1$ and when $v = 4$?

Since we solved the problem for $v = 1$, take a few moments on your own to solve the problem for $v = 4$.

You should find that $s'(4) \approx 2.16$.

Conclusion

- The *derivative* of $f(x)$ at $x=b$ is the *slope of the line tangent to* $f(x)$ at $x=b$. This is the limit of the slopes of the secant lines.
- The derivative of f at $x=b$ is formally defined to be

Definition: *(The derivative at a point)*

$$f'(b)=\lim_{x\to b}\frac{f(x)-f(b)}{x-b}$$

where $f'(b)$ is the symbol used for the derivative of f at $x=b$.

- If the derivative of a function exists at a point, we say that the function is differentiable at that point.
- We can estimate this value graphically or numerically.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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