Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Properties of the Limit

We use limits a great deal in Calculus. As a result, we need to analyze the limit carefully and identify any properties that might prove useful. We will in fact apply some of the properties discussed here in future modules.

Defining the problem

We would like to evaluate the following limits *without* numerical or graphical approximation, and *without* verifying ϵ and δ.

- $\lim_{x\to 0} c$, where c is any real number.
- $\lim_{h\to 0}(e^x f(h))$, where x is any real number and f is any function in terms of h *and not* x.
- $\lim_{h\to 0}\frac{f(x)}{g(x)}$ when we know $\lim_{h\to 0} f(x)$ and $\lim_{h\to 0} g(x)$.
- Other algebraic combinations of f and g.

SAGE worksheets

You should open the SAGElet "Limits: ϵ-δ".

A word on continuity

If you have not studied the module on continuity yet, you can ignore this slide. Otherwise, read on.

You instructor may have chosen to discuss with you the topic of continuity before covering this material. If so, you may notice that some of the properties we will give seem redundant with what was discussed in that slide. This is not really true. We gave in that module a theorem that certain functions were continuous. We did not prove the theorem, however, and with good reason: the proof of that theorem relies on properties of the limit described below.

A word on proofs

If you have not studied the module on limits using ϵ-δ, you can ignore any slide that talks about ϵ-δ. Many instructors choose not to dwell on this approach.

It is possible to prove all of the properties that we describe in this module using an ϵ-δ argument. We will prove only a few of them this way; the others we will state without any argument at all.

First properties

The first properties that we describe deal with a constant or a constant multiple.

Theorem: Let c be a constant with respect to x. Then for any $a \in \mathbb{R}$ and for any function $f(x)$,

- $\lim_{x\to a} c = c$; and
- $\lim_{x\to a} f(x) = c \lim_{x\to a} f(x)$, if the latter limit exists.

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For an example of the first property,

$$\lim_{x\to 3} 4 = 4.$$

Module: Properties of the limit 5

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For an example of the second property,

$$\lim_{x\to 1} 4x^2 = 4 \lim_{x\to 1} x^2 = 4 \cdot 1 = 4.$$

Module: Properties of the limit 5

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- $\lim_{x\to a} c = c$; and
- $\lim_{x\to a} f(x) = c \lim_{x\to a} f(x)$, if the latter limit exists.

Another example of the second property that we will use in a future module is

$$\lim_{h\to 0} (e^x f(h)) = e^x \lim_{h\to 0} f(h)$$

where f is a function in terms of h *and not* x. Notice that e^x depends only on h, so it is constant *with respect to* h.

Proof of the limit of a constant: concept

It is relatively easy to prove these two properties using an ϵ-δ argument.

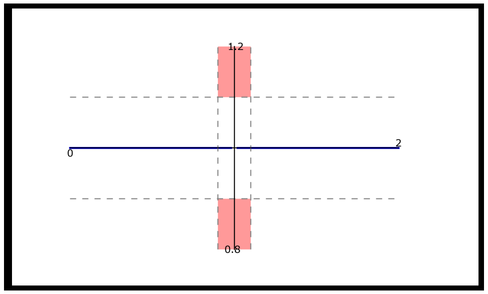
We remind you that the ϵ-δ argument comes from the precise definition of a limit:

Definition: *(Precise definition of the limit)*
$\lim_{x\to a} f(x) = L$ if

- for all $\epsilon > 0$,
- there exists $\delta > 0$
- such that for all x satisfying $|x - a| < \delta$ (except maybe $x = a$),
- $|f(x) - L| < \epsilon$.

Proof of the limit of a constant: concept

It is relatively easy to prove these two properties using an ϵ-δ argument. In the SAGE worksheet “Limits: ϵ-δ”, set f to be the constant 1. (To do this, you must type `lambda x: 1` in the `f` box. This strange notation is due to a bug which ought to be resolved soon.) You should see the following plot.



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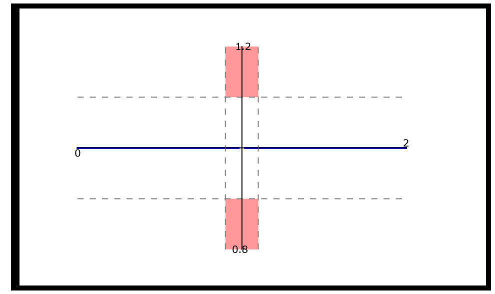

Look carefully at the plot. Notice that the graph of f does not pass through the pink regions. This means that you have found an appropriate value of δ to guarantee that the distance between L and $f(x)$ is smaller than ϵ, satisfying the definition of the limit.

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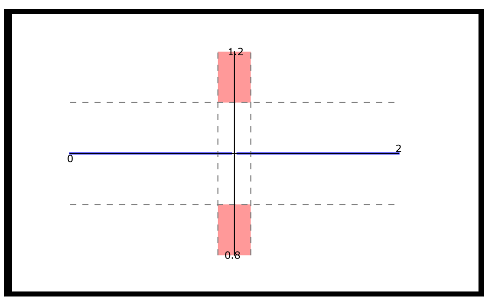

Now look more carefully. Experiment with different values of δ. Can you find any value of δ such that $f(x)$ passes through the pink regions?

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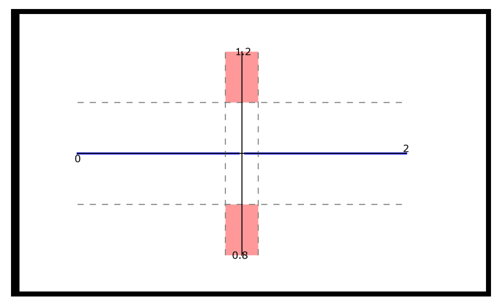

Now look more carefully. Experiment with different values of δ. Can you find any value of δ such that the graph of $f(x)$ passes through the pink regions?

You should not be able to find such a value. In fact, $|f(x)-L| = |1-1| = 0 < \epsilon$ *regardless of the value of* ϵ.

Proof of the limit of a constant

We can generalize this argument to show that for any constant function $f(x)=c$ and for any real number a, $\lim_{x\to a} f(x)=c$. Let ϵ be any positive real number, and then

$$|f(x)-L|=|c-c|=0<\epsilon.$$

You can choose *any* value of δ and satisfy the precise definition of the limit.

Proof of the limit of a constant multiple: concept

To determine how to prove the limit of a constant multiple, we'll take a similar approach, and try an example function and an example constant. Let $f(x)=x^2$, $a=1$, and $c=-2$. We want to see if we can show that

$$\lim_{x\to 1}(-2x^2)=-2\lim_{x\to 1}x^2.$$

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A glance at the graph suggests that $\lim_{x\to 1}x^2=1$. Look again at the SAGElet, and start with $f(x)=x^2$. Using $\epsilon=0.1$, find δ satisfying the definition of the limit.

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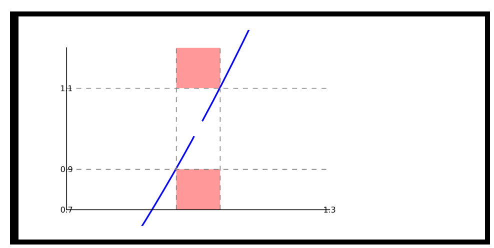

You should find that $\delta\approx 0.05$ works. (In fact the correct answer is $\delta=-1+\sqrt{1.1}\approx 0.0488$.)

Proof of the limit of a constant multiple: concept

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$$\lim_{x\to 1}(-2x^2)=-2\lim_{x\to 1}x^2.$$

Using $\delta=0.05$ guarantees that $|f(x)-1|<0.1$.

Now try the SAGElet with $cf(x)=-2x^2$. Keep $\delta=0.05$. What value of ϵ works with this δ to guarantees that $|cf(x)-cL|<\epsilon$? (Note $cL=-2\cdot 1=-2$.)

Module: Properties of the limit 8

Proof of the limit of a constant multiple: concept

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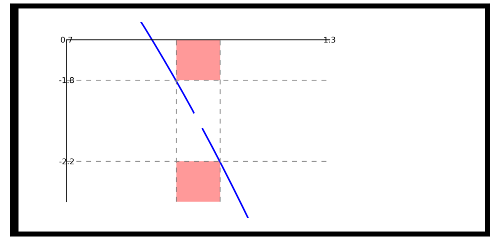


You should find that $\epsilon=0.2$. This is twice the value of the previous epsilon.

Module: Properties of the limit 8

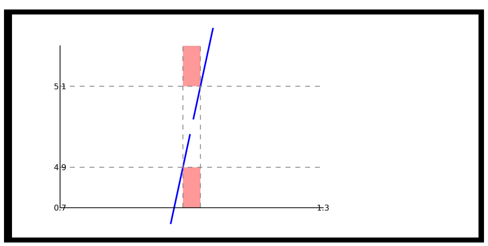
Proof of the limit of a constant multiple: concept

To determine how to prove the limit of a constant multiple, we'll take a similar approach, and try an example function and an example constant. Let $f(x)=x$, $a=1$, and $c=5$. We want to see if we can show that

$$\lim_{x\to 1}5x=5\lim_{x\to 1}x.$$

Using $\delta=0.05$ guarantees that $|f(x)-1|<0.1$.

Now try the SAGElet with $cf(x)=-2x^2$. Keep $\delta=0.05$. What value of ϵ works with this δ to guarantees that $|cf(x)-cL|<\epsilon$? (Note $cL=-2\cdot 1=-2$.)



You should find that $\epsilon=0.2$. This is twice the value of the previous epsilon.

Perhaps the δ that makes ϵ work for $cf(x)$, is the same as the δ that makes $\epsilon/|c|$ work for $f(x)$.

Proof of the limit of a constant multiple

We try this in a proof. We want to show that $\lim_{x\to a}cf(x)=c\lim_{x\to a}f(x)$, if the latter limit exists. Assume that the latter limit *does* exist; that is, $\lim_{x\to a}f(x)=L$.

We need to show that $\lim_{x\to a}cf(x)=cL$.

Proof of the limit of a constant multiple

We try this in a proof. We want to show that $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$, if the latter limit exists. Assume that the latter limit *does* exist; that is, $\lim_{x\to a} f(x) = L$.

We need to show that $\lim_{x\to a} cf(x) = cL$.

If $c = 0$ then $cf(x) = 0$, and we are talking about the limit of a constant. We have already discussed this case, so we can assume that $c \neq 0$.

Module: Properties of the limit 9

Proof of the limit of a constant multiple

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We need to show that $\lim_{x\to a} cf(x) = cL$.

By the definition of a limit, we must show that for every positive ϵ we can find a positive δ satisfying

$$|x-a| < \delta \quad \Longrightarrow \quad |cf(x) - cL| < \epsilon.$$

Let $\epsilon' = \epsilon/|c|$; this is a positive number. Since L is the limit of $f(x)$ as $x \to a$, the precise definition of a limit tells us that there exists δ such that

$$|x-a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon'.$$

Module: Properties of the limit 9

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$$|x-a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon'.$$

Then

$$|cf(x) - cL| = |c(f(x) - L)| = |c| \cdot |f(x) - L| < |c| \cdot \epsilon' = \epsilon.$$

Module: Properties of the limit 9

Proof of the limit of a constant multiple

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Then

$$|cf(x) - cL| = |c(f(x) - L)| = |c| \cdot |f(x) - L| < |c| \cdot \epsilon' = \epsilon.$$

We have found δ such that if $|x-a| < \delta$, then $|cf(x) - cL| < \epsilon$. We have proved the second part of the theorem:

$$\lim_{x\to a} cf(x) = cL = c\lim_{x\to a} f(x).$$

The limit of x

Our next property is very convenient.

Theorem: For all $a \in \mathbb{R}$,
$$\lim_{x\to a} x = a.$$

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The graph of $f(x) = x$ gives credence to the theorem.

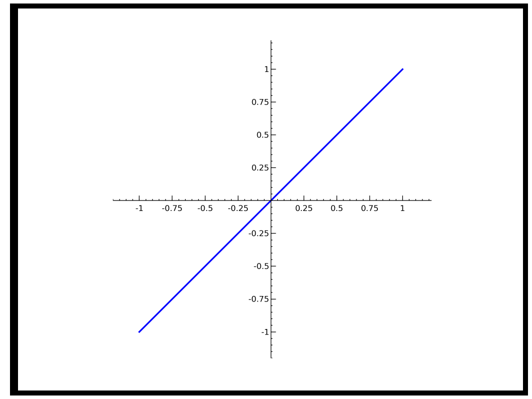

Figure 1: If you approach $x = 0$ from both sides, it looks as if the limit is 0.

The limit of x

Our next property is very convenient.

Theorem: For all $a \in \mathbb{R}$,
$$\lim_{x\to a} x = a.$$

Let's experiment a little with the SAGElet to see how δ corresponds to ϵ. Change `f` to `x`, and pick any x-value you like. What is the largest value of δ that satisfies the precise definition of the limit for $\epsilon = 0.1$?

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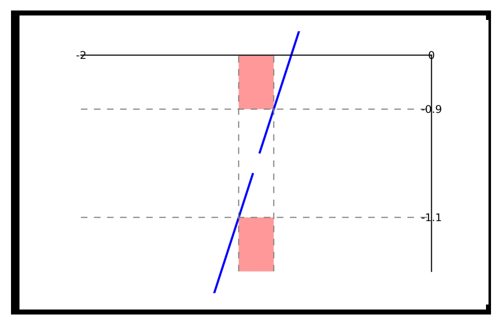

It looks here as if $\delta = 0.1$ works. What if we change ϵ to 0.01?

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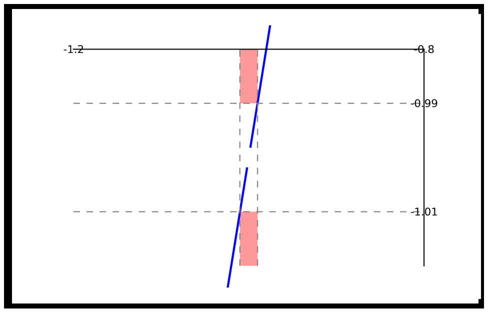

It looks here as if $\delta = 0.01$ works. What if we change ϵ to 0.5?

The limit of x

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Theorem: For all $a \in \mathbb{R}$,
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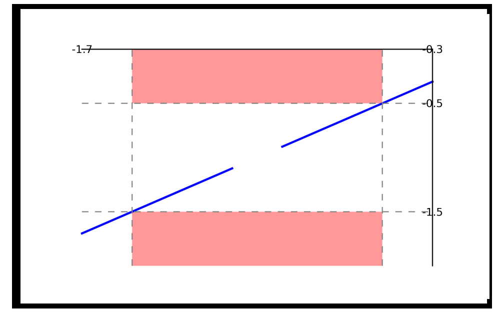

It looks here as if $\delta = 0.5$ works.

The limit of x

Our next property is very convenient.

Theorem: For all $a \in \mathbb{R}$,
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In each case, δ had the same value as ϵ. Let's try that in a proof.

The limit of x

Our next property is very convenient.

Theorem: For all $a \in \mathbb{R}$,
$$\lim_{x\to a} x = a.$$

In each case, δ had the same value as ϵ. Let's try that in a proof.

Let ϵ be any positive real number, and set $\delta = \epsilon$. Assume that $|x - a| < \epsilon$. Then
$$|f(x) - L| = |x - a|.$$
By hypothesis, $|x - a| < \delta = \epsilon$, so
$$|f(x) - L| < \epsilon.$$
We have thus proved the theorem.

More properties

Here are almost all the remaining properties. We do not prove them.

Theorem: Let $a \in \mathbb{R}$, and f and g functions such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

- The limit of a sum is the sum of the limits:
$$\lim_{x\to a}(f(x) \pm g(x)) = L + M.$$
- The limit of a product is the product of the limits:
$$\lim_{x\to a}(f(x)g(x)) = LM.$$
- The limit of a quotient is the quotient of the limits:
If $M \neq 0$, then $\lim_{x\to a}(f(x)/g(x)) = L/M$.
- If $n \in \mathbb{N}$ then $\lim_{x\to a} x^n = a^n$ and $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{L}$.
- If $n \in \mathbb{N}$ then $\lim_{x\to a} f(x)^n = L^n$ and $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$, *provided that* $\sqrt[n]{L}$ exists.
- If f is a *polynomial* or *rational expression* and a is in the domain of f, then you can substitute the value $x = a$ into f to find the limit:
$$\lim_{x\to a} f(x) = f(a).$$

Examples

Suppose we have two probes whose value at x is modeled by the functions $f(x)$ and $g(x)$, and we know that $\lim_{x\to 3} f(x) = -2.5$ and $\lim_{x\to 3} g(x) = 0$. Then

$$\lim_{x\to 3}(f(x) + g(x)) = -2.5$$
$$\lim_{x\to 3}\frac{g(x)}{f(x)} = 0$$
$$\lim_{x\to 3}\sqrt[4]{-f(x)} = \sqrt[4]{-\lim_{x\to 3} f(x)} \approx 1.2574.$$

We have to be careful with division by zero. In particular, we *cannot* say that

$$\lim_{x\to 3}\frac{f(x)}{g(x)}$$

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Examples

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does not exist.

Limits *can* exist even when the denominator of a quotient approaches zero. This is not the same as division by zero, because we are considering $x \to a$ and *not* $x = a$. You saw an example of this in a previous module, when we showed that

$$\lim_{x\to 0}\frac{x^2 - x}{x} = -1$$

even though $\lim_{x\to 0} x = 0$!

Examples

We have to be careful with division by zero. In particular, we *cannot* say that

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does not exist.

Algebraically, we can show this as follows:

$$\lim_{x\to 0}\frac{x^2 - x}{x} = \lim_{x\to 0}\frac{\cancel{x}(x-1)}{\cancel{x}}$$

(Factored common x in numerator.)

Examples

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does not exist.

Algebraically, we can show this as follows:

$$\begin{aligned}\lim_{x\to 0}\frac{x^2-x}{x} &= \lim_{x\to 0}\frac{x(x-1)}{\cancel{x}}\\ &= \lim_{x\to 0}(x-1)\\ &= \lim_{x\to 0}x - \lim_{x\to 0}(-1)\\ &= -1.\end{aligned}$$

(Notice that we used the properties of
(1) the sum of a limit,
(2) the limit of a constant, and
(3) the limit of x.)

Module: Properties of the limit 12

Examples

We have to be careful with division by zero. In particular, we *cannot* say that

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does not exist.

On the other hand, limits do not *have* to exist, either. If $f(x)=\frac{|x|}{x}$, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{-x}{x} = -1\ldots$$

(If $x\to 0^-$, the value of x is negative, so $|x|=-x$.
This puzzles people who read it the first time,
so think about it carefully—it is *not* a typo!)

Module: Properties of the limit 12

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but

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{x}{x} = 1.$$

Module: Properties of the limit 12

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So

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

and there is no two-sided limit.

When division by zero appears in a limit, *further investigation is needed*.

Conclusion

- We identified (and used!) a number of properties of the limit. These properties will be useful in future modules.
- We proved some properties using the precise definition of the limit (ϵ-δ). The others can also be proved, of course.
- If we are using the quotient property of limits and encounter division by zero, then further investigation is needed. A useful tool for such investigation is factoring. Another useful tool is analyzing the one-sided limits.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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