Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Derivatives of trigonometric functions

In this module we will determine more efficient ways of computing the derivatives of *trigonometric functions*. We will use the precise definition of the derivative to explain these methods.

Defining the problem

The precise definition of the derivative tells us that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

It is often tedious to compute the derivative using the precise definition. We would like a shorter way of computing the derivative for trigonometric functions.

You saw in the module on experimental derivatives that the derivative of $\sin x$ appears to be $\cos x$. In this module we argue that this is in fact the case, and discuss as well the derivatives of the other trigonometric functions $\cos x$, $\tan x$, $\cot(x)$, $\sec x$, $\csc x$.

SAGE worksheets

You will need a blank SAGE worksheet for this module.

The derivative of $\sin x$

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$$\sin(x+h) = \sin x \cos h + \sin h \cos x.$$

We'll rewrite the expression above using this fact.

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Next we'll collect expressions containing $\sin x$.

The derivative of $\sin x$

Let $f(x) = \sin x$. We want to compute the derivative of $f(x)$.

From the definition, we know that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \sin h \cos x}{h}. \end{aligned}$$

At this point we will apply some algebraic machinery. The limit would look a lot easier if we could remove the terms containing x. Unfortunately, we can't factor them, so we have to try something different. We'll try the property of limits that *if* the limit of two functions g and k exist, then the limit of their sum is the sum of their limits. That is…

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$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \sin h \cos x}{h} \\ &\stackrel{?}{=} \lim_{h \to} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h \cos x}{h} \\ &\stackrel{?}{=} \sin x \cdot \lim_{h \to} \frac{(\cos h - 1)}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}. \end{aligned}$$

The symbol $\stackrel{?}{=}$ indicates that we don't know the two sides are equal, because we don't know that these limits exist.

(We also used the constant multiple property of limits: $\sin x$ and $\cos x$ are constant with respect to a limit on h.))

Do these limits exist?

We will not prove that these limits have values; we will simply approximate them. Turn to SAGE now and estimate the limits

$$\lim_{h\to}\frac{(\cos h-1)}{h}=?$$
$$\lim_{h\to 0}\frac{\sin h}{h}=?.$$

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$$\lim_{h\to 0}\frac{\sin h}{h}=?.$$

(*Hint:* Use either a numerical table or a graph to approximate the limit.)

Do these limits exist?

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$$\lim_{h\to}\frac{(\cos h-1)}{h}=0$$
$$\lim_{h\to 0}\frac{\sin h}{h}=1.$$

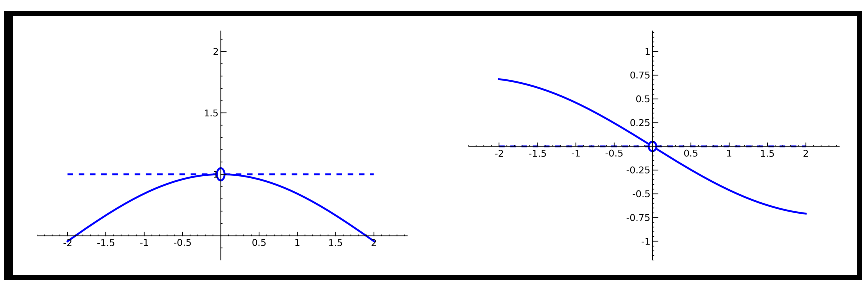

Figure 1: Plots of $\sin h/h$ (left) and $(\cos h-1)/h$ (right).

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$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\sin h\cos x}{h} \\ &= \lim_{h\to}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\sin h\cos x}{h} \\ &= \sin x\cdot\lim_{h\to}\frac{(\cos h-1)}{h}+\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}. \end{aligned}$$

Now we know that the limits exist, so we can apply the limit properties without qualms.

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$$\begin{aligned} f'(x) &= \sin x \cdot \lim_{h\to} \frac{(\cos h - 1)}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1. \end{aligned}$$

The derivative of $\sin x$

Thus

Theorem: The derivative of $\sin x$ is $\cos x$.

This confirms our intuition from the experimentation.

The derivative of $\cos x$

Now let's turn to the derivative of $\cos x$.

Experimenting with the Function Factory and Tangent Line Guesser SAGElets suggests that the derivative of $\cos x$ is $-\sin x$. The precise definition gives us

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}.$$

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(Trig property.)

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(Collected $\cos x$.)

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(Limit properties. We argued earlier that these limits exist.)

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The derivative of $\cos x$

Thus

Theorem: The derivative of $\cos x$ is $-\sin x$.

The derivatives of other trig functions

Other trig functions can be derived using

- the derivatives of $\sin x$ and $\cos x$, and
- the properties of derivatives.

For example, $\tan x = (\sin x)/(\cos x)$, so its derivative is (from the quotient rule)

$$\frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

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(Pythagorean identity.)

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The derivative of other trig functions

Similar arguments will give the derivatives of other trig functions. We omit the details and summarize here.

Theorem: *(The derivatives of basic trig functions)*

- The derivative of $\sin x$ is $\cos x$.
- The derivative of $\cos x$ is $-\sin x$.
- The derivative of $\tan x$ is $\sec^2 x$.
- The derivative of $\cot x$ is $-\csc^2 x$.
- The derivative of $\sec x$ is $\sec x \tan x$.
- The derivative of $\csc x$ is $-\csc x \cot x$.

You should commit these to memory.

Example 1

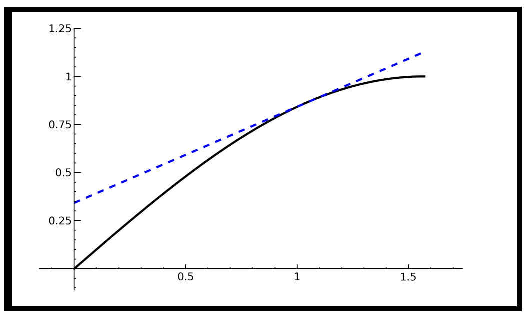
The properties of derivatives allow us to say that if $f(x) = 3\tan x - 2\cos x$ then

$$f'(x) = 3\sec^2 x + 2\sin x.$$

Example 2

If we want to find the equation of the line tangent to $\sin x$ at $x = \pi/3$, its slope is $\cos(\pi/3)$ and so the line is

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$
$$y = \frac{x}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}.$$



Example 3

Since $1 \approx \pi/3$, we can compute a linear approximation of $\sin 1$ using the tangent line at $x = \pi/3$.

$$\begin{aligned}\sin 1 \approx L(1) &= \cos\left(\frac{\pi}{3}\right)\left(1 - \frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\\ &= \frac{1}{2}\left(1 - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\\ &= \frac{3\left(1+\sqrt{3}\right) - \pi}{6}\\ &\approx 0.8424.\end{aligned}$$

In fact, $\sin 1 \approx 0.8415$.

Conclusion

- We found a shortcut for the derivatives of trigonometric functions.

Theorem: *(The derivatives of basic trig functions)*

- The derivative of $\sin x$ is $\cos x$.
- The derivative of $\cos x$ is $-\sin x$.
- The derivative of $\tan x$ is $\sec^2 x$.
- The derivative of $\cot x$ is $-\csc^2 x$.
- The derivative of $\sec x$ is $\sec x \tan x$.
- The derivative of $\csc x$ is $-\csc x \cot x$.

- To find these shortcut, we used the precise definition of the derivative, argued the values of limits using geometry, and used properties of the derivative.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

Department of Mathematics at
The University of
Southern Mississippi