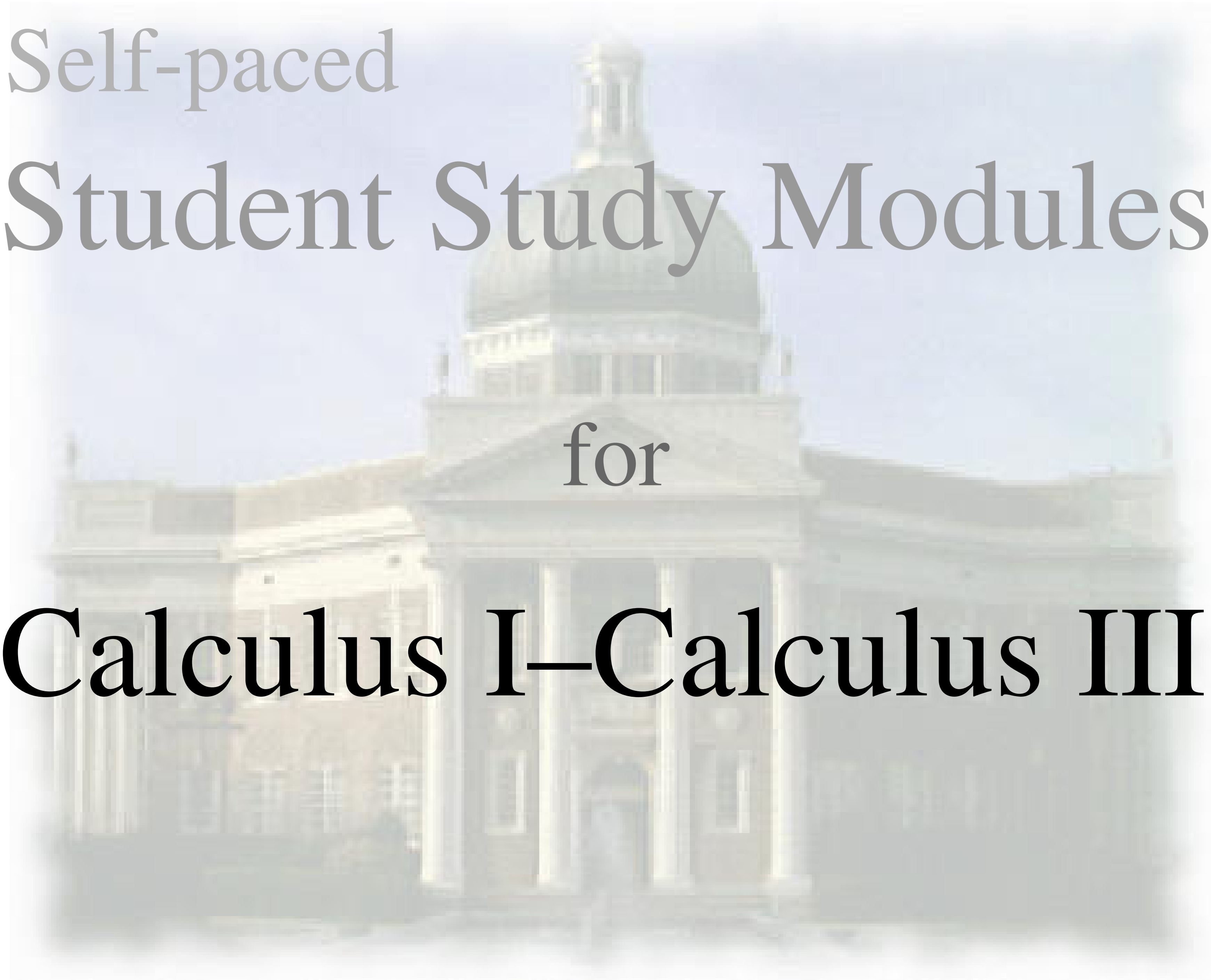


Self-paced Student Study Modules for Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Implicit Differentiation

So far we have studied derivatives only in the context of functions of one variable such as $y = f(x)$. There are cases where we would like to compute the derivative for a *relation* that is not a function, or that cannot easily be written in the form $y = f(x)$. In these circumstances, it can still make sense to compute a derivative—all we need is that the limit of the slopes of the secant lines exists—but since we cannot isolate y from x, we need a more general method than the ones used heretofore. We call this method *implicit differentiation*.

Defining the problem

Question: How can we find the derivative at any point of the following relations *quickly*?

- $x^2 + y^2 = 1$
- $x^2 e^{xy} = x + y$
- $\arcsin x$, $arctanx$, $arcsecx$

SAGE worksheets

You will need a blank SAGE worksheet for this module.

Implicit Differentiation

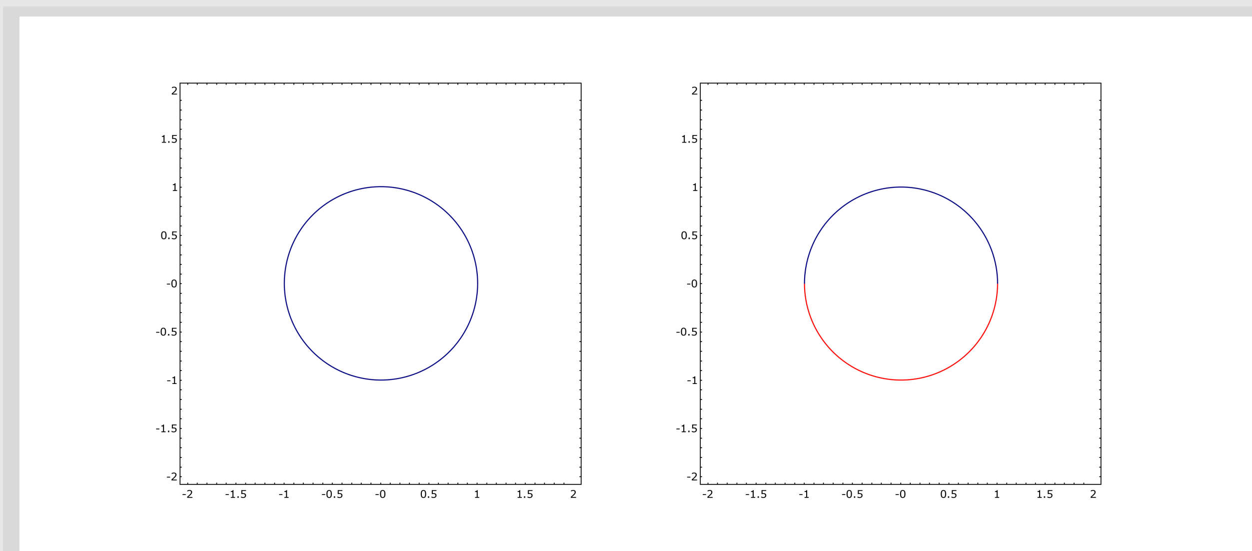
If we look back at the first example,

$$x^2 + y^2 = 1,$$

we notice that it is possible to solve for y explicitly:

$$y = \pm\sqrt{1 - x^2}.$$

This is not a function, but we can choose either the positive or negative branch to compute the derivative at a given point.



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$$y = \sqrt{1 - x^2} = (1 - x^2)^{1/2}$$

$$y' = \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x).$$

(Don't forget the Chain Rule.)

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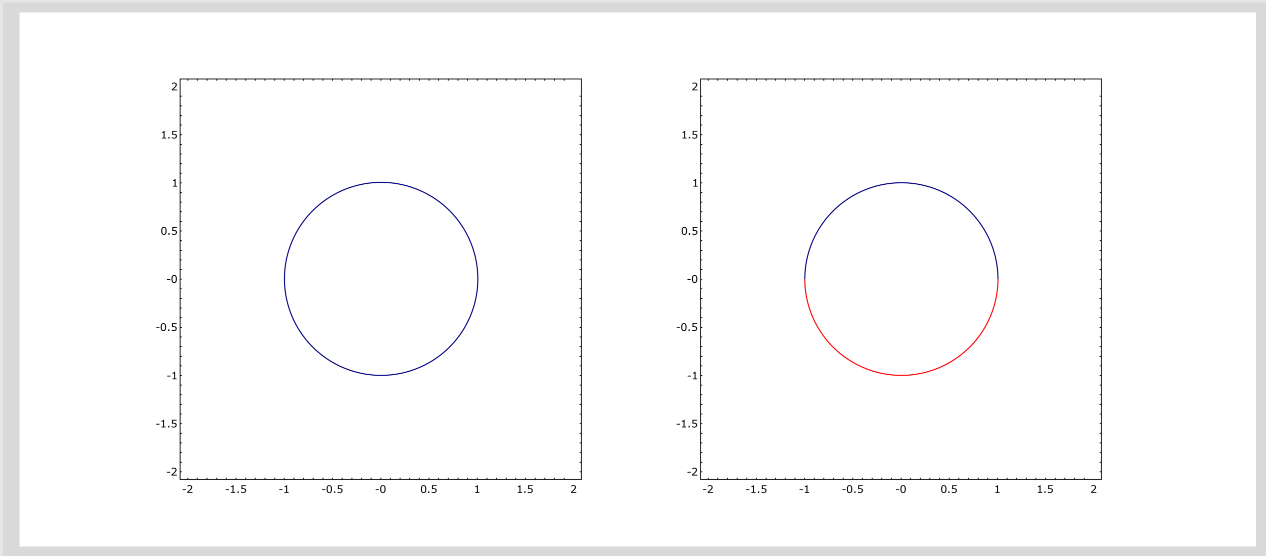
Either way, we found

$$y' = \mp\frac{x}{\sqrt{1 - x^2}}.$$

Implicit Differentiation

Even though y is not a function of x in the circle C, we can find functions y_1 and y_2 such that C is the union of the points described by two functions.



Thus the relation $x^2 + y^2 = 1$ defines y as a function of x *locally*. We can treat y as a function that depends on x, and differentiate using the Chain Rule.

First Example

We demonstrate this using the first example.

$$x^2 + y^2 = 1.$$

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$$x^2 + y^2 = 1$$

$$2x + 2y \cdot y' = 0.$$

(Since y is a function of x, the Chain Rule tells us that the derivative of y^2 is $2y \cdot y'$.)

First Example

We demonstrate this using the first example.

$$
\begin{aligned}
x^2+y^2 &= 1 \\
2x+2y\cdot y' &= 0 \\
y' &= -\frac{x}{y}.
\end{aligned}
$$

(Solve for y'.)

First Example

We demonstrate this using the first example.

$$\begin{aligned} x^2+y^2 &= 1 \\ 2x+2y\cdot y' &= 0 \\ y' &= -\frac{x}{y}. \end{aligned}$$

We illustrate this by computing the slopes of lines tangent to the circle at various points.

First Example

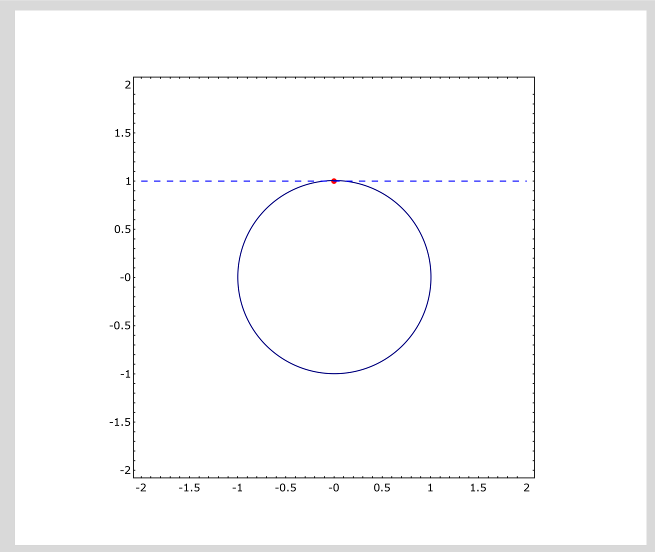
We demonstrate this using the first example.

$$y' = -\frac{x}{y}.$$

We illustrate this by computing the slopes of lines tangent to the circle at various points.

The point $(0,1)$ is on the curve, since $0^2 + 1^2 = 1$. At this point,

$$y' = -0/1 = 0.$$



First Example

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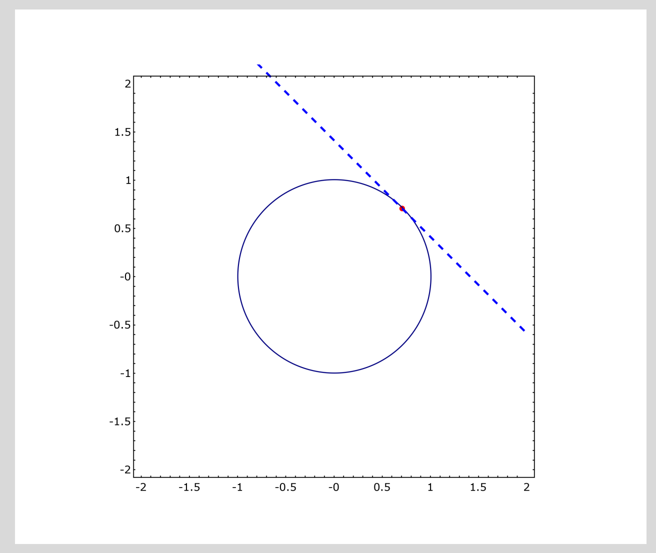
$$y' = -\frac{x}{y}.$$

We illustrate this by computing the slopes of lines tangent to the circle at various points.

The point $(\sqrt{2}/2, \sqrt{2}/2)$ is on the curve, since $\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$. At this point,

$$y' = -\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1.$$



First Example

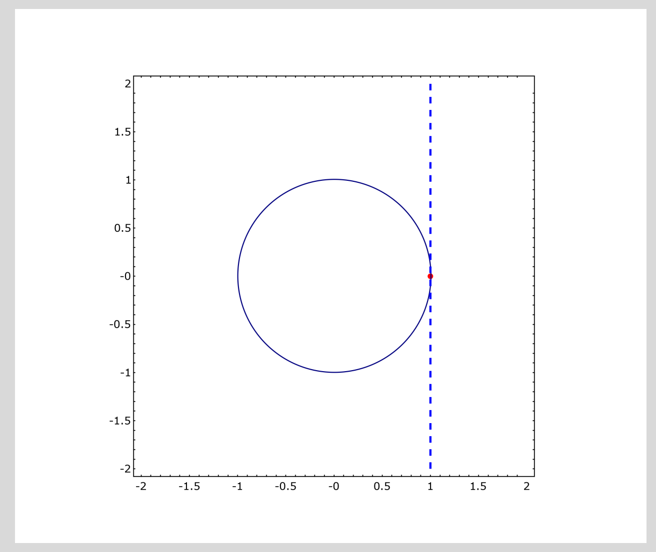
We demonstrate this using the first example.

$$y' = -\frac{x}{y}.$$

We illustrate this by computing the slopes of lines tangent to the circle at various points.

The point $(1,0)$ is on the curve, since $1^2 + 0^2 = 1$. At this point,

$$y' = -\frac{1}{0}, \text{ corresponding to a vertical line.}$$



Implicit Plots

It isn't always easy to view the graph of a relation. If you can't separate y and x, you can still plot it in SAGE using the `implicit_plot()` command. For example, the graphs of the circles were generated using these commands:

```
circleplot = implicit_plot(x^2+y^2-1,(x,-2,2),(y,-2,2),cmap='jet',plot_points=500)
circleplot.show(aspect_ratio=1)
upperplot = implicit_plot(x^2+y^2-1,(x,-2,2),(y,0,2),cmap='jet',plot_points=500)
lowerplot = implicit_plot(x^2+y^2-1,(x,-2,2),(y,-2,0),cmap='hsv',plot_points=500)
(upperplot+lowerplot).show(aspect_ratio=1)
```

(Try it!)

The command takes as its first argument an expression `exp` which corresponds to the equation `exp` $=0$. We want to plot $x^2+y^2=1$, which corresponds to $x^2+y^2-1=0$, so `exp` $=x^2+y^2-1$. The second and third arguments are a range of x- and y-values over which the expression should be graphed. The third argument corresponds to a color (the usual colors don't apply in the currect version of SAGE; we advise you to stick with `'jet'` (blue), `'hsv'` (red), and `summer` (green)). The final argument should be included in order to obtain an accurate graph from SAGE; if the number is too small, the graph will not be accurate.

Second Example

We can also differentiate the second example, where it is not possible to isolate y.

$$x^2 e^{xy} = x + y.$$

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$$x^2 e^{xy} = x + y$$
$$2x e^{xy} + x^2 e^{xy} \cdot (y + x y') = 1 + y'.$$

(Use the Product Rule and the Chain Rule. We only write y' when we take the derivative of y.)

Second Example

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$$x^2 e^{xy} = x + y$$

$$2xe^{xy} + x^2 e^{xy} \cdot (y + xy') = 1 + y'$$

$$2xe^{xy} + x^2 y e^{xy} + x^3 e^{xy} \cdot y' = 1 + y'.$$

(Distribution.)

Second Example

We can also differentiate the second example, where it is not possible to isolate y.

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$$2x e^{xy} + x^2 y e^{xy} + x^3 e^{xy} \cdot y' = 1 + y'$$

$$\left(x^3 e^{xy} - 1\right) y' = 1 - e^{xy}\left(x^2 + 2x\right).$$

(Everything with y' on one side;
everything without it to the other.)

Second Example

We can also differentiate the second example, where it is not possible to isolate y.

$$\begin{aligned} x^2 e^{xy} &= x + y \\ 2x e^{xy} + x^2 e^{xy} \cdot (y + xy') &= 1 + y' \\ 2x e^{xy} + x^2 y e^{xy} + x^3 e^{xy} \cdot y' &= 1 + y' \\ \left(x^3 e^{xy} - 1\right) y' &= 1 - e^{xy}\left(x^2 + 2x\right) \\ y' &= \frac{1 - e^{xy}\left(x^2 + 2x\right)}{x^3 e^{xy} - 1}. \end{aligned}$$

(Solve for y'.)

Third Example

The third example asks us to find the derivatives of the inverse functions of $\sin x$, $\tan x$, and $\sec x$. The derivatives of these functions have an interesting form. Implicit differentation allows us to find these derivatives. Start with $\arcsin x$:.

$$y = \arcsin x.$$

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$$y = \arcsin x$$

$$\sin y = x.$$

(Properties of inverse functions.)

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$$\sin y = x$$

$$\cos y \cdot y' = 1.$$

(Differentiated both sides with respect to x.)

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$$\cos y \cdot y' = 1$$

$$y' = \frac{1}{\cos y}.$$

(Solve for y'.)

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$$y' = \frac{1}{\cos y}$$

$$y' = \frac{1}{\cos(\arcsin x)}.$$

(Substitute $y = \arcsin x$.)

Third Example

The third example asks us to find the derivatives of the inverse functions of $\sin x$, $\tan x$, and $\sec x$. The derivatives of these functions have an interesting form. Implicit differentation allows us to find these derivatives. Start with $\arcsin x$:.

$$y' = \frac{1}{\cos(\arcsin x)}$$

$$y' = \frac{1}{\sqrt{1-x^2}}.$$

(This is a property learned in precalculus; we explain it briefly. We want to know $\cos\alpha$ where $\alpha = \arcsin x$. Thus $x = \sin\alpha$, so we can draw a right triangle with hypotenuse of length 1 that has an angle of measure α and a leg opposite α of length x. The length of the side adjacent to α will then be $\sqrt{1-x^2}$.)

Third Example

We have the following.

Theorem: *(Derivatives of inverse trig functions)*

- If $f(x) = \arcsin x$, then $f'(x) = \frac{1}{\sqrt{1-x^2}}$.
- If $f(x) = \arctan x$, then $f'(x) = \frac{1}{1+x^2}$.
- If $f(x) = \operatorname{arcsec} x$, then $f'(x) = \frac{1}{x\sqrt{x^2-1}}$.

(You should check on your own that the other two are correct.)

Conclusion

- *Implicit differentiation* allows us to compute the derivative in cases where y may not be a function of x.
- Implicit differentiation uses the chain rule: whenever you have to take the derivative of y, you multiply by y'.
- Implicit differentiation allowed us to determine the derivatives of three new functions:

Theorem: *(Derivatives of inverse trig functions)*

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- If $f(x) = \arctan x$, then $f'(x) = \frac{1}{1+x^2}$.
- If $f(x) = \operatorname{arcsec} x$, then $f'(x) = \frac{1}{x\sqrt{x^2-1}}$.

You should remember these derivatives.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the **⟨Print⟩** icon, and then saving or printing the pdf file.

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