Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

The Squeeze Theorem

Sometimes, the limit of a function f is difficult to evaluate using direct methods, but we can compare its values to those of other functions. Using the other functions as a guide, we can sometimes determine the limit of f.

Sample problem

The problem: Evaluate

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right).$$

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How would you start the problem?

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How would you start the problem?

Take a look at the graph of $x \sin\left(\frac{1}{x}\right)$.

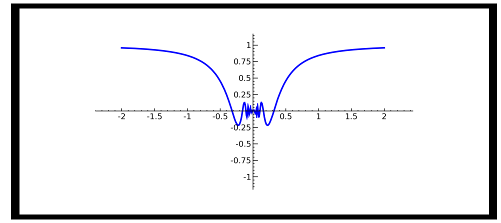

Figure 1: We see that f *appears* to approach zero as x approaches zero.

The problem with geometry

On the other hand, we can't always rely on geometric intuition. For instance, $(x+0.01)\sin\left(\frac{1}{x}\right)$ also appears to approach zero as x approaches zero:

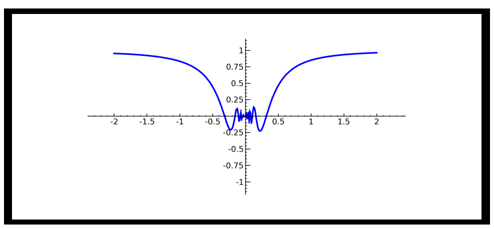

Figure 2: We see that $(x+0.01)\sin\left(\frac{1}{x}\right)$ *appears* to approach zero as x approaches zero.

More of a problem with geometry

If we zoom in on the x-axis, we see that there might *not* be a limit at $x=0$.

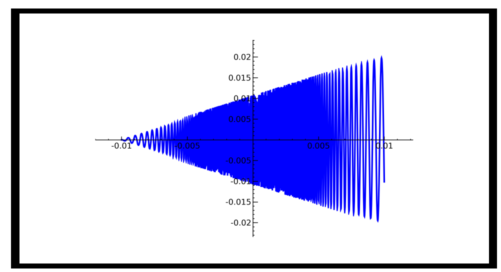

Figure 3: Here $(x+0.01)\sin\left(\frac{1}{x}\right)$ *appears* to wiggle wildly as x approaches zero.

In this case there is no limit at $x=0$.

Squeezing a function

In many cases, a function is trapped between two other functions. Another way of saying this is that it is squeezed between two functions. For example, $f(x) = x\sin\left(\frac{1}{x}\right)$ is trapped between the two functions $g(x) = x$ and $h(x) = -x$.

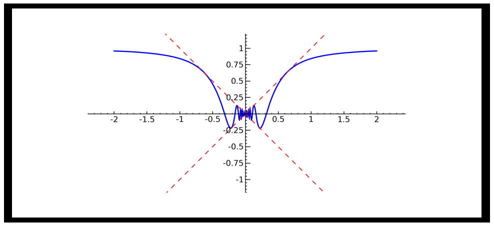

Figure 4: The dashed red lines are $y = \pm x$.

In this case, we have $h(x) \le f(x) \le g(x)$ for all x.

How do we know it is squeezed?

How do we know that the function is squeezed? All that we saw there was a graph; zooming in sufficiently might show that something strange is going on.

Module: The Squeeze Theorem 6

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We can use simple facts from trigonometry to analyze the function. Start with

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \quad \forall x \in \mathbb{R}.$$

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$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \quad \forall x \in \mathbb{R}.$$

We want to look at $x\sin(\frac{1}{x})$, not $\sin(\frac{1}{x})$, so multiply all sides by x. If $x \ge 0$, we have

$$-x \le x\sin\left(\frac{1}{x}\right) \le 1 \quad \forall x \ge 0.$$

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We want to look at $x\sin(\frac{1}{x})$, not $\sin(\frac{1}{x})$, so multiply all sides by x. If $x < 0$, we change the direction of the inequalities, and

$$-x \geq x\sin\left(\frac{1}{x}\right) \geq 1 \quad \forall x \geq 0.$$

How does this help with the limits?

At this point we describe a well-known result in mathematics.

Theorem: *(The Squeeze Theorem)*
Let f, g, and h be functions on $I = (a, b)$ and $c \in I$. If

- $f(x)$ lies between $g(x)$ and $h(x)$ for all $x \in I$, and
- $L = \lim_{x\to c} g(x) = \lim_{x\to c} h(x)$,

then $\lim_{x\to c} f(x) = L$.

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Read this carefully. Try to identify which functions from the previous slide correspond to f, g, h, c, and L.

Applying the theorem to our example

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In our example, we want to show that $\lim_{x\to 0} x\sin(1/x) = 0$. We want to use the fact that $x\sin(1/x)$ always lies between x and $-x$. These values correspond ($\mapsto$) in the following way to the terms of the Squeeze Theorem:

$$\begin{aligned} f(x) &\mapsto x\sin(1/x), \\ g(x) &\mapsto x, \\ h(x) &\mapsto -x, \\ c &\mapsto 0, \\ L &\mapsto 0. \end{aligned}$$

Since $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(-x) = 0$, what can we conclude about $\lim_{x\to 0} x\sin(1/x)$?

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$$\begin{aligned} f(x) &\mapsto x\sin(1/x), \\ g(x) &\mapsto x, \\ h(x) &\mapsto -x, \\ c &\mapsto 0, \\ L &\mapsto 0. \end{aligned}$$

By the Squeeze Theorem, $\lim_{x\to 0} x\sin(1/x)=0$.

The bad example

What about that function with an infinite wiggle? In this case,

$$x+0.01\geq (x+0.01)\sin\left(\frac{1}{x}\right)\geq -(x+0.01).$$

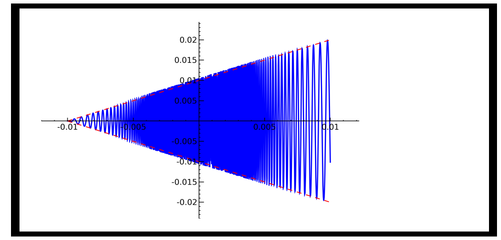

Figure 5: The dashed red lines correspond to $y=\pm(x+0.01)$. It may look as if the function crosses those lines, but it does not.

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Here, as $x\to 0$,

$$\lim_{x\to 0}(x+0.01)=0.01,$$

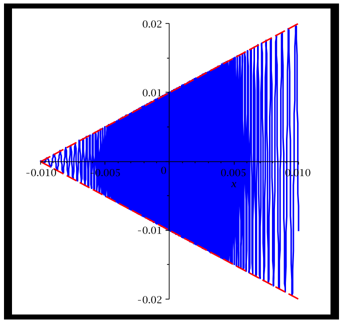
whereas

$$\lim_{x\to 0}(x-0.01)=-0.01.$$

Since the limits are unequal, the hypothesis of the Squeeze Theorem are unsatisfied. We cannot conclude anything about $\lim\limits_{x\to 0}(x+0.01)\sin\left(\frac{1}{x}\right)$.

The bad example: putting it together

Geometrically, we see this in graph of the function $y = (x + 0.01)\sin(1/x)$:



The limits "look" as if they should be $\lim_{x\to 0}(x+0.01) = 0.01$, and $\lim_{x\to 0}(x-0.01) = -0.01$, as explained on the previous slide.

Finding functions to squeeze another

The Squeeze Theorem allows you to apply your understanding of a limit in many places. With the sine and cosine functions, you can usually Start with

$$-1 \leq \sin(\text{anything}) \leq 1, \quad \text{and} \quad -1 \leq \cos(\text{anything}) \leq 1.$$

Then build this relationship up to the point that we are investigating, as we did on page **??**.

The process of estimation

In finding functions $g(x)$ and $h(x)$ such that some known function $f(x)$ satisfies

$$g(x) \leq f(x) \leq h(x), \qquad \forall x,$$

we have managed to bound the behaviour of $f(x)$. In this case $g(x)$ is a lower bound and $h(x)$ is an upper bound for $f(x)$.

The process of replacing a known function with an estimate, usally a lower or upper bound, or both, is know as estimation. A large part of mathematical analysis consists of trying to find estimates that desribe the behaviour of very complicated functions.

Conclusion

- The Squeeze Theorem allows us to evaluate easily the values of many limits that are otherwise difficult.
- To use the Squeeze Theorem, you need to identify upper and lower bounds for a function. Those upper and lower bounds must have the same limit at the point in question.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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