Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

The Mean Value Theorem

You already know that the slope of the tangent line is the limit of the slopes of the secant lines. The Mean Value Theorem gives us another, very useful connection between the slopes of secant lines and the slopes of tangent lines. We make use of the Mean Value Theorem in future modules to justify some applications.

Sample problems

Question:

- Let $f(x) = \sin(\frac{\pi}{2}x)$. Is there an x value for which $f'(x) = 1$?
- Suppose that f and g have the same derivative. What can I conclude about f and g: are they equal? if not, how close are they?

SAGE worksheets

For this module you will not need a SAGE worksheet, but you can use a blank one to draw some of the graphs we consider.

The Mean Value Theorem

The Mean Value Theorem (MVT) gives us a new and interesting connection between the slope of a function's secant lines and the slope of a tangent line.

Theorem: *(The Mean Value Theorem)*
Suppose that f is a function that is continuous on $[a,b]$ and differentiable on (a,b). There exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

What exactly is this saying?

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What exactly is this saying?
Notice that

$$\frac{f(b)-f(a)}{b-a}$$

is the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$. On the other hand, $f'(c)$ is the slope of the tangent line at $x = c$. If MVT's hypotheses about f, a, and b are satisfied, then we can find a point c where the slope of the tangent line is the same as the slope of the secant line.

Example 1

We'll illustrate MVT using the first example. Recall the question:

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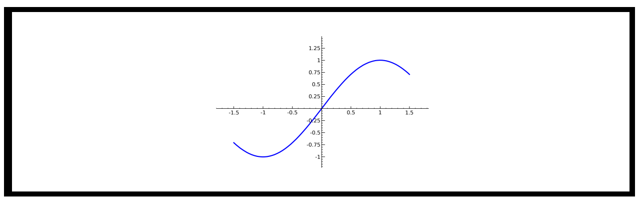
We could answer "YES!!!" if we could find a, b satisfying MVT's hypotheses *and* having a secant line whose slope is 1.

Such a, b exist! If $a = 1$ and $b = -1$, then

$$\frac{f(b) - f(a)}{b - a} = \frac{\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)}{1-(-1)} = \frac{1-(-1)}{2} = 1.$$

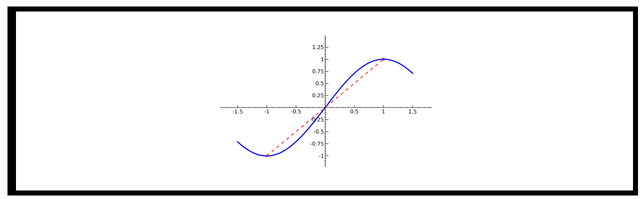
Example 1 illustrated

How about some pictures? The plot of $\sin(\frac{\pi}{2}x)$ looks like this:



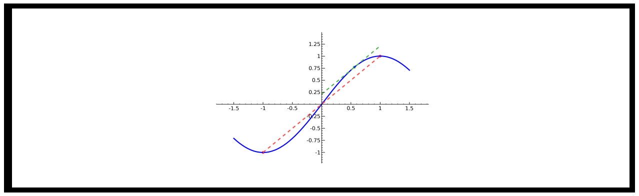
Example 1 illustrated

The secant line connect $x = -1$ and $x = 1$ has slope 1:



Example 1 illustrated

MVT tells us that a point $x = c$ exists whose tangent line has the slope of this secant line:



Module: The Mean Value Theorem 5

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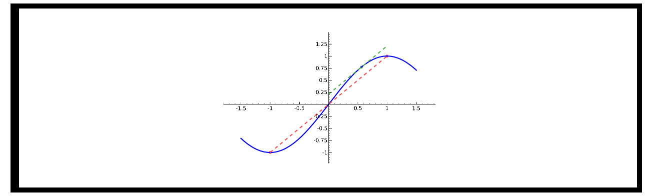

In this case, we can use calculus to show that

$$c = \frac{2}{\pi} \arccos \frac{2}{\pi}.$$

Finding the value of is not always easy, necessary—or even possible!—so MVT allows us to decide whether such a point exists without making that effort.

Rolle's Theorem

MVT can be proved using a similar fact called Rolle's Theorem:

> **Theorem:** *(The Mean Value Theorem)*
> Suppose that f is a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that
> $$f'(c) = 0.$$

You can see that Rolle's Theorem is a special case of MVT, since the slope of the secant line between $x = a$ and $x = b$ is zero. We will not prove either MVT or Rolle's Theorem, but your textbook has a proof, and your instructor can provide one if necessary.

Example 2

Recall Question 2:

> **Question:** Suppose that f and g have the same derivative. What can I conclude about f and g: are they equal? if not, how close are they?

The answer to this question is quite important actually:

> **Theorem:** If $f'(x) = g'(x)$ for all $x \in (a, b)$, then f and g differ only by a constant; that is,
> $$f(x) = g(x) + c \qquad \exists c \in \mathbb{R}.$$

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Why is this true? Suppose that $f'(x) = g'(x)$ for all $x \in (a,b)$. First, notice that
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Why is this true? Suppose that $f'(x) = g'(x)$ for all $x \in (a,b)$. First, notice that
$$f'(x) - g'(x) = 0$$
for all $x \in (a,b)$. Let $F = f - g$; by substitution
$$F'(x) = 0$$
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Why is this true? ... Let $F = f - g$; by substitution
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for all $x \in (a,b)$. Let x_1, x_2 be any two numbers in (a,b); by the Mean Value Theorem there exists $c \in (x_1, x_2)$ such that
$$F'(c) = \frac{F(x_1) - F(x_2)}{x_1 - x_2}.$$

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However, $F'(x) = 0$ for *all* $x \in (x_1, x_2) \subset (a,b)$, so $F'(c) = 0$. The equation above thus simplifies to
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For example, since the derivative of $\sin x$ is $\cos x$, we know now that *every* function whose derivatrive is $\cos x$ has the form

$$\sin x + C$$

where C is a constant.

Conclusion

In this module, we discussed:

- The Mean Value Theorem (MVT):

  Theorem: Suppose that f is a function that is continuous on $[a,b]$ and differentiable on (a,b). There exists $c \in (a,b)$ such that

  $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- As a consequence of MVT

  Theorem: If $f'(x) = g'(x)$ for all $x \in (a,b)$, then f and g differ only by a constant; that is,

  $$f(x) = g(x) + c \qquad \exists c \in \mathbb{R}.$$

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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