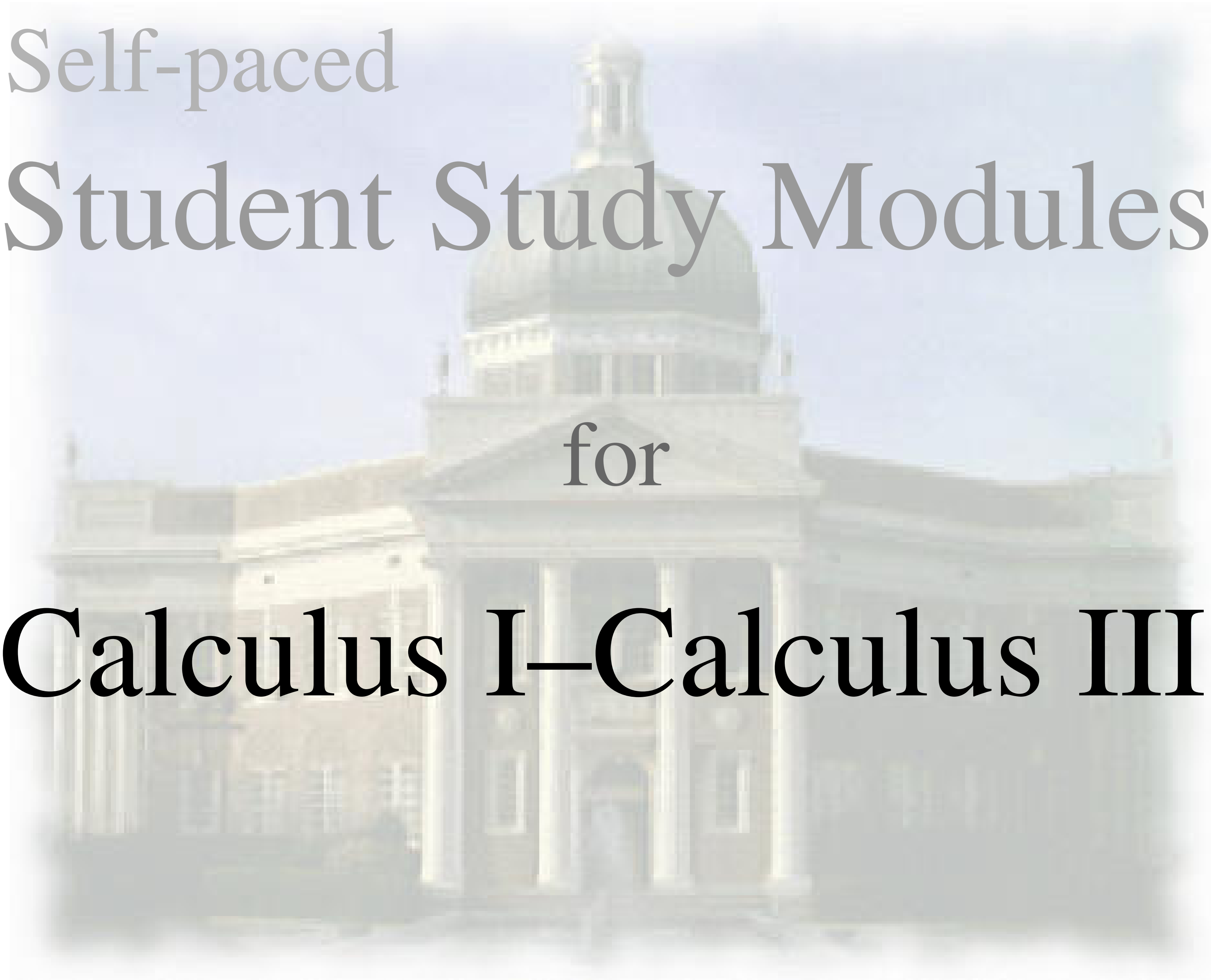

Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages **⟨Next⟩**, backward **⟨Prev⟩**, or view all the slides in this tutorial **⟨Index⟩**.
- The **⟨Back to Calc I⟩** button returns you to the course home page.
- A full symbolic algebra package **⟨Sage⟩** is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text **⟨CalcText⟩** provides a quick search of basic calculus topics.
- You can get help from Google Calculus **⟨GoogleCalc⟩**.
- A monochrome copy of this module is suitable for printing **⟨Print⟩**.

When all else fails, feel free to contact your instructor.

Derivatives of logarithmic functions

In this module we determine the derivative of the logarithmic functions $\ln x$. We also use it to find the derivatives of more difficult functions.

Defining the problem

Question: What are the derivatives of the following functions?

- $y = \ln x$
- $y = x^{\sqrt{x}}$

The derivative of $\ln x$

We start with the derivative of $\ln x$. Before showing explicitly what its derivative is, let's take an educated guess using the SAGElets "Tangent Line Guesser" and "Function Factory".

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Using the Tangent Line Guesser, estimate the value of the derivative of $\ln x$ at the following values of x.

x	$\frac{1}{2}$	1	2	5	10
$(\ln x)'$					

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(SAGE may report an error after you change the value of f in the SAGElet to `ln(x)`. This is expected, because a is initially `0` and $\ln 0$ is undefined. Change the value of a and the error should disappear.)

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Once you have some values, plug them into the Function Factory.

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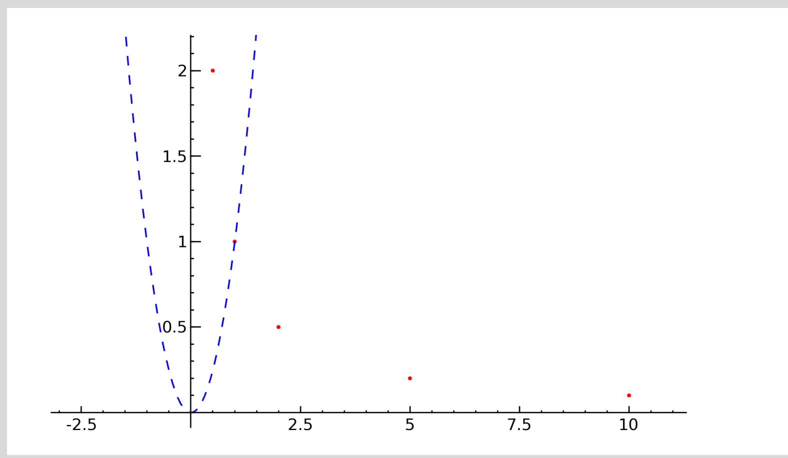
Once you have some values, plug them into the Function Factory. Your function should look like the one below. You should recognize it; modify f to match it.



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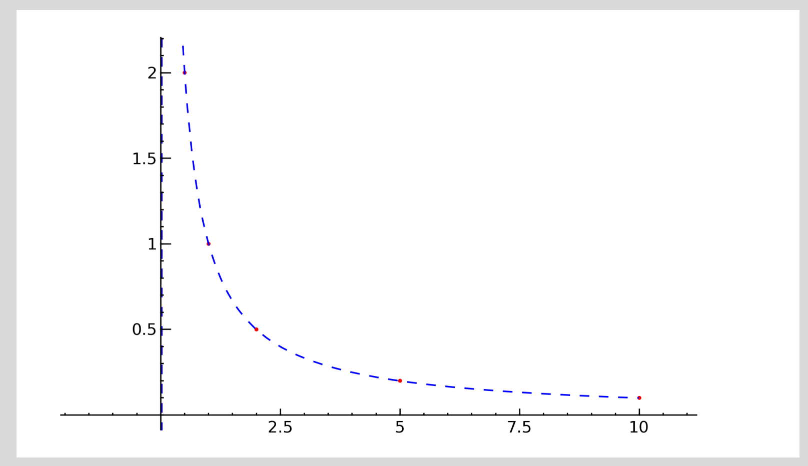
Using the Tangent Line Guesser, estimate the value of the derivative of $\ln x$ at the following values of x.

x	$\frac{1}{2}$	1	2	5	10
$(\ln x)'$	2	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$

Once you have some values, plug them into the Function Factory. Your function should look like the one below. You should recognize it; modify f to match it.



This looks like $1/x$. But is it?

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$$y = \ln x$$
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$$e^y = x.$$

(The exponential function and the logarithm are inverse functions.)

The derivative of $\ln x$

To find the derivative of $\ln x$, we will use implicit differentiation.

$$y = \ln x$$
$$e^y = e^{\ln x}$$
$$e^y = x$$
$$e^y \cdot y' = 1.$$

(Took the derivative of both sides *with respect to* x; implicit differentation tells us that the derivative of e^y with respect to x is $e^y \cdot y'$.)

The derivative of $\ln x$

To find the derivative of $\ln x$, we will use implicit differentiation.

$$\begin{aligned} y &= \ln x \\ e^y &= e^{\ln x} \\ e^y &= x \\ e^y \cdot y' &= 1 \\ y' &= e^{-y}. \end{aligned}$$

(Multiplied both sides by e^{-y}.)

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To find the derivative of $\ln x$, we will use implicit differentiation.

$$
\begin{aligned}
y &= \ln x \\
e^y &= e^{\ln x} \\
e^y &= x \\
e^y \cdot y' &= 1 \\
y' &= e^{-y} \\
y' &= e^{-\ln x}.
\end{aligned}
$$

(Substituted $y = \ln x$—see first line.)

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(Property of logarithms.)

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e^y \cdot y' &= 1 \\
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y' &= e^{-\ln x} \\
y' &= e^{\ln x^{-1}} \\
y' &= x^{-1} \\
y' &= \frac{1}{x}.
\end{aligned}
$$

(Property of exponents.)

The derivative of $\ln x$

The preceding argument confirms our initial guess.

Theorem: The derivative of $\ln x$ is $1/x$.

Logarithmic differentiation

Once we know the derivative of the logarithm, we can use it to find derivatives of functions where *the variable is in both the base and the exponent*. We call this technique logarithmic differentiation.

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$$\begin{aligned} y &= x^{\sqrt{x}} \\ \ln y &= \ln x^{\sqrt{x}} \\ \ln y &= \sqrt{x}\ln x \\ \frac{1}{y}\cdot y' &= \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x}. \end{aligned}$$

(Took the derivative of both sides. By implicit differentiation and the derivative of the logarithm, $(\ln y)' = \frac{1}{y}\cdot y'$.
The derivative of the right hand side uses the product rule.)

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(Multiplied y to both sides.)

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(Substitued $y = x^{\sqrt{x}}$—see first line.)

Derivatives of other logarithms

For the derivative of a different logarithmic base, we can reason from the base conversion rule for logarithms that

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a}\frac{1}{x} = \frac{1}{x \ln a}.$$

Thus

Theorem: For any $a \in \mathbb{R}$ where $a > 0$ and $a \neq 1$, the derivative of $\log_a x$ is $\frac{1}{x \ln a}$.

Conclusion

- We determined the derivative of $\ln x$ both geometrically and precisely, using implicit differentiation.
- The derivative $\ln x$ allows us to compute the derivatives of functions with variables in both the base and the exponent using logarithmic differentiation.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the **⟨Print⟩** icon, and then saving or printing the pdf file.

Department of Mathematics at
The University of
Southern Mississippi