MAT 167 Calculus Solution

- **Problem 4**

  Determine on which domain

$$g(x) = \begin{cases} (x^2-4)/(x+2) & \text{if } x < 0 \\ 5 & \text{if } x = 0 \\ (x^3+2x^2-2x-4)/(x^2+4x+4) & \text{if } x > 0 \end{cases}$$

  is continuous. Can the function be redefined to have a greater domain of continuity?

  Solution

  - If $x < 0$ then:

$$g(x) = \frac{(x^2-4)}{(x+2)} = \frac{(x-2)(x+2)}{(x+2)}$$

    This function is not continous since at $x = -2$ the function is not defined at $x = -2$. However, since

$$\lim_{x \to -2^-} g(x) = \frac{(x-2)(x+2)}{(x+2)} = -4$$

$$\lim_{x \to -2^+} g(x) = \frac{(x-2)(x+2)}{(x+2)} = -4$$

    the limit from the left equals the limit from the right, so that if we define

$$\tilde{g}(x) = (x-2)$$

    then $\tilde{g}(x) = x - 2$ and $\tilde{g}(x) = g(x)$ except at $x = -2$ and $\tilde{g}$ is continuous on $(-\infty, 0)$.

  - If $x > 0$ then:

$$g(x) = (x^3+2x^2-2x-4)/(x^2+4x+4) = \frac{(x^2-2)(x+2)}{(x+2)^2} = \frac{x^2-2}{x+2}$$

    and $g(x)$ has no singularities on the domain $(0, \infty)$ since $x = -2$ where the denominator vanishes is outside the domain of definition of this branch of the function. Thus

$$g(x) = (x^3+2x^2-2x-4)/(x^2+4x+4) = \frac{x^2-2}{x+2}$$

    are the same functions and $g(x) = (x^2-2)/(x+2)$ is continous on $(0, \infty)$.

– If $x = 0$, then consider

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x - 2 = -2,$$

and

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{x^2-2}{x+2} = -1.$$

Thus $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$ and $g(x)$ is not continous at $x = 0$. This is not a removeable discontinuity since we cannot redefine $g(0)$ in a manner which makes the limit from the left equal the limit from the right and equal to the function values $g(0)$.

From this the conclusion is that $g(x)$ as defined, is continuous on the intervals $(-\infty, 2) \bigcup (2, 0) \bigcup (0, \infty)$.

– If we define

$$\tilde{g}(x) = \begin{cases} x-2 & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ (x^2-2)/(x+2) & \text{if } x > 0 \end{cases}$$

then $\tilde{g}(x) = g(x)$ except at $x = -2$ and $x = 0$ and $\tilde{g}$ is continuous on $(-\infty, 0] \bigcup (0, \infty)$.

– If we define

$$\overline{g}(x) = \begin{cases} x-2 & \text{if } x < 0 \\ -1 & \text{if } x = 0 \\ (x^2-2)/(x+2) & \text{if } x > 0 \end{cases}$$

then $\overline{g}(x) = g(x)$ except at $x = -2$ and $x = 0$ and $\overline{g}$ is continuous on $(-\infty, 0) \bigcup [0, \infty)$.

There is no way to redefine g to be continuous on $(-\infty, \infty)$.

- Explain the behavior of $f(x) = a_0e^x + a_1x + a_2x^2$ as $x \to \infty$ and $x \to -\infty$.

Solution

Consider the case when $x \to \infty$:

– Observe that as $x \to \infty$ the term e^x dominates any polynomial, that is, for all x sufficiently large

$$e^x > x^n.$$

Thus for large x the behaviour of $f(x)$ will be determined by the behaviour of a_0x^2. Thus, if $a_0 > 0$ then $f(x) \to \infty$ if $a_0 < 0$ then $f(x) \to -\infty$.

- If $a_0 = 0$ then the term $a_0 e^x$ vanishes and the behaviour of the function as $x \to \infty$ is determined by the term $a_2 x^2$. Thus if $a_2 > 0$ then $f(x) \to \infty$ and if $a_2 < 0$ then $f(x) \to -\infty$.
- If $a_0 = 0$ and $a_2 = 0$ then the terms $a_0 e^x$ and $a_2 x^2$ vanish and the behaviour of the function as $x \to \infty$ is determined by the term $a_1 x$. Thus if $a_1 > 0$ then $f(x) \to \infty$ and if $a_1 < 0$ then $f(x) \to -\infty$.
- If all of the $a_i = 0$. Then the function is identically zero for all x.

Consider the case when $x \to -\infty$:

- Observe that as $x \to -\infty$ the term e^x vanishes that is,

  $$\lim_{x \to -\infty} e^x \to 0.$$

  Thus for large negative x the behaviour of $f(x)$ will be determined by the behaviour of $a_2 x^2$, hence if $a_2 > 0$ then $f(x) \to \infty$, and if $a_2 < 0$ then $f(x) \to -\infty$.
- If $a_2 = 0$ then the term $a_2 x^2$ vanishes and the behaviour of the function as $x \to -\infty$ is determined by the term $a_1 x$. Thus if $a_1 > 0$ then $f(x) \to -\infty$ and if $a_1 < 0$ then $f(x) \to \infty$.
- If $a_1 = a_2 = 0$ then the function behaves like $a_0 e^x$, that is, as $x \to -\infty$, then $f(x) \to 0$.
- If all of the $a_i = 0$, then the function is identically zero for all x.