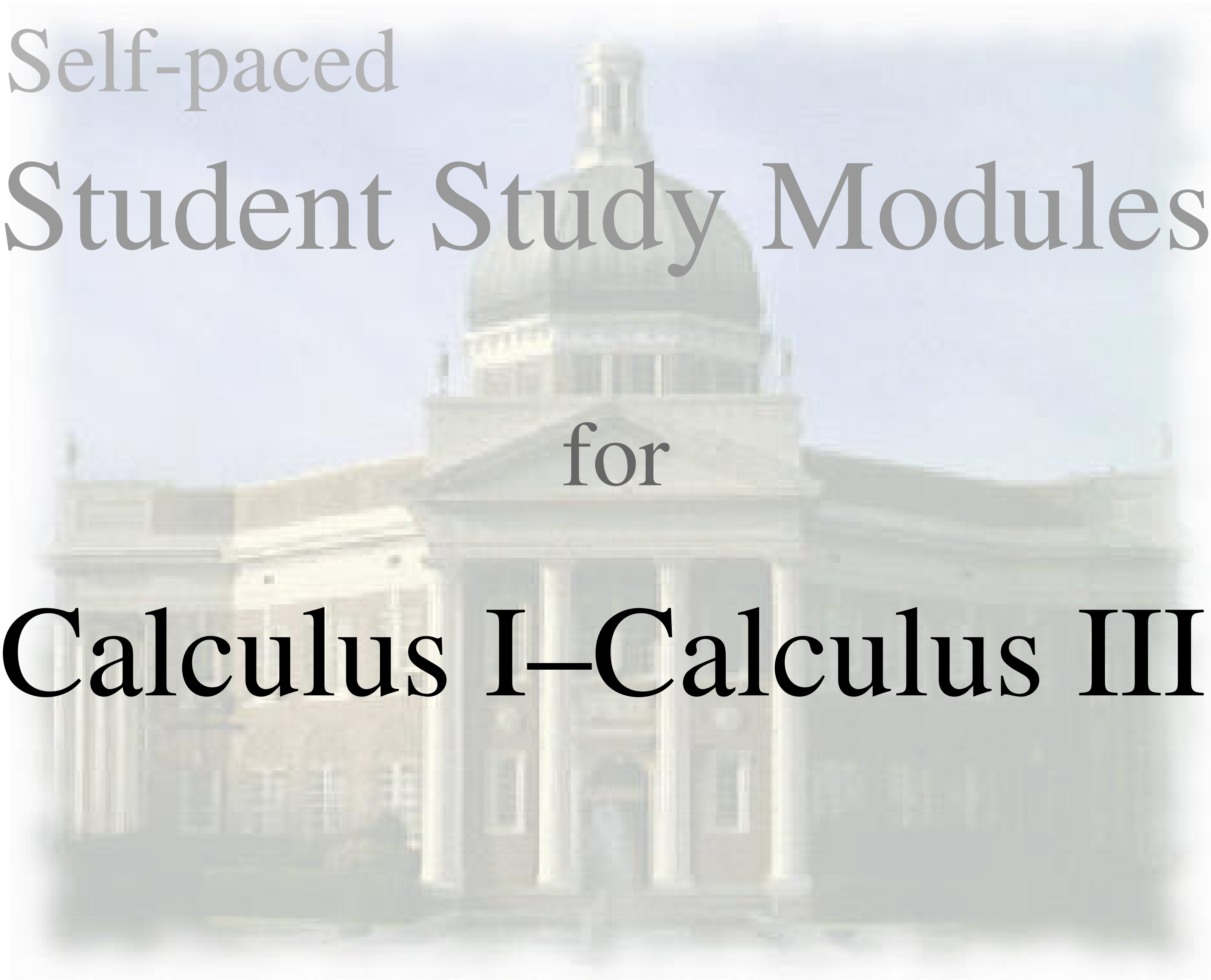


Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Overview

The graph of a function can help us visualize how a function behaves. Graphing calculators and computer algebra systems allow us to see many aspects of a graph easily, but before using a graphing calculator or a computer algebra system you need to know *where* to look for interesting features. For this we need a tool called *curve sketching*.

Sample problems

Question:

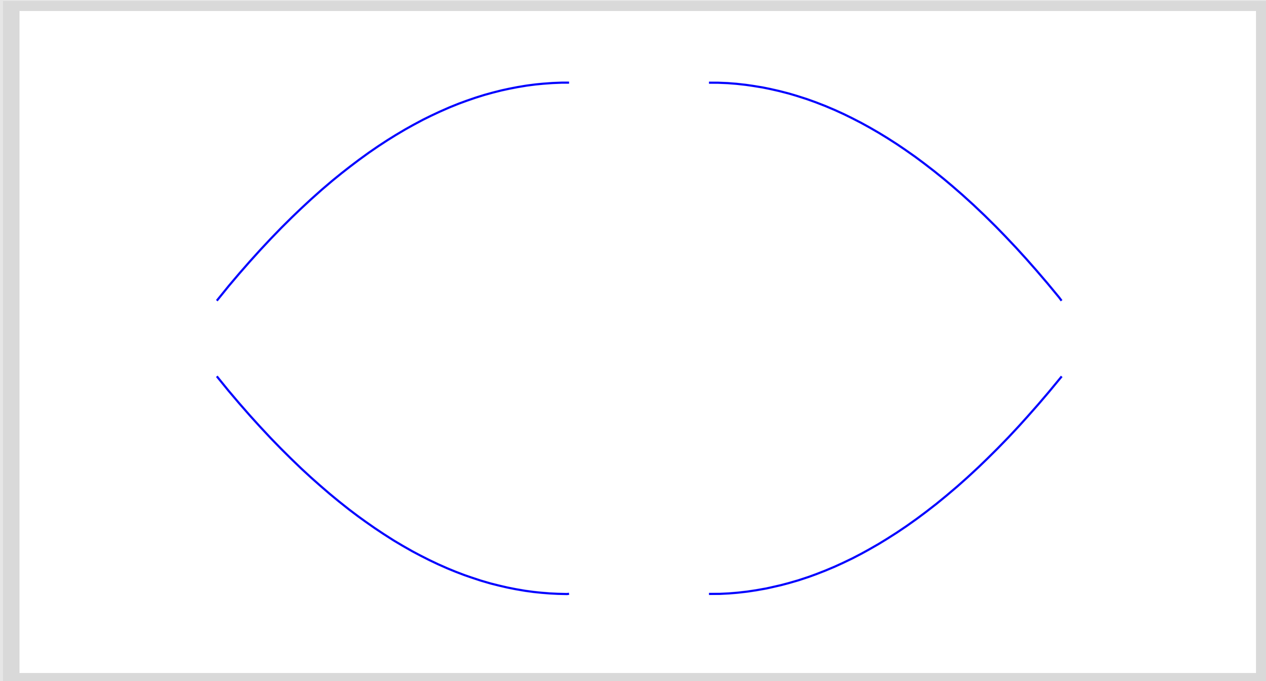
- Make a sketch of the curve $x^3 + 2x - 1$.
- Sketch the curve of $-10x^3 + \frac{x}{10} + \frac{1}{10}$, labeling every point where
  - the curve starts and stops increasing to decreasing; and
  - the curve's slope starts and stops increasing and decreasing.

SAGE worksheets

You will not need a SAGE worksheet for this module.

Four components of curves

We are interested in four general shapes that a curve can take:



We can assemble these four shapes, along with lines and asymptotes, to provide a general idea of the shape of any curve.

Four components of curves

What is interesting about those four shapes? We need to discuss two basic ideas.

Four components of curves

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Definition: *(Increasing and decreasing)*

- We say that a function is *increasing* on the interval $I=(a,b)$ when for any $x_1, x_2 \in I$, if $x_2 > x_1$, then $f(x_2) > f(x_1)$.
- We say that a function is *decreasing* on the interval $I=(a,b)$ when for any $x_1, x_2 \in I$, if $x_2 > x_1$, then $f(x_2) < f(x_1)$.

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Which of the four basic shapes correspond to *increasing* functions?

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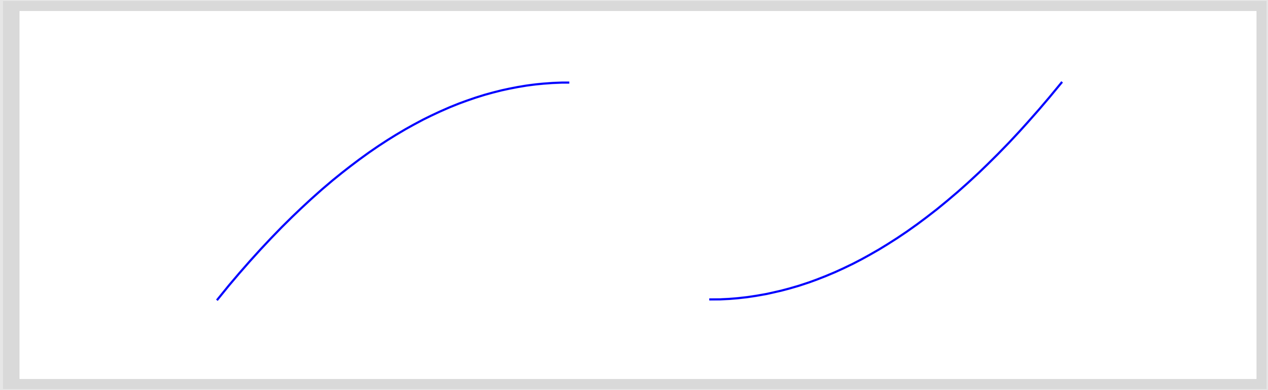

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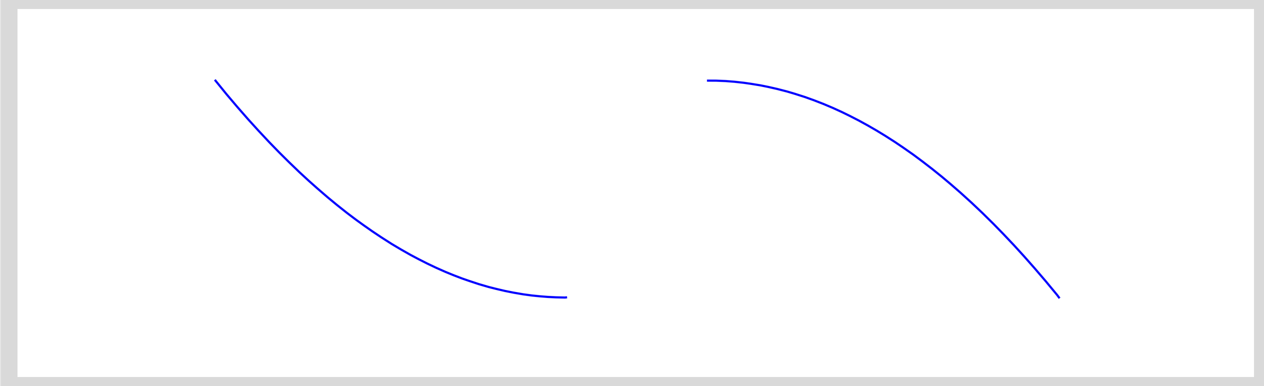
Which of the four basic shapes correspond to *decreasing* functions?



Derivatives and the direction of a function

Finding extrema allows us to determine when a function is increasing or decreasing. To review,

- Find all places where $f'(x)=0$ or $f'(x)$ does not exist. These x values are called critical points.
- Find the y values at each critical points *and* at the endpoints.
- The smallest y value found is the minimum, and the largest y value found is the maximum.

Once we locate the extrema, continuity tells us that we can connect them with one of the curves give on page 3.

- An *increasing* curve has a minimum on the left and a maximum on the right; and
- a *decreasing* curve has a maximum on the left and a minimum on the right.

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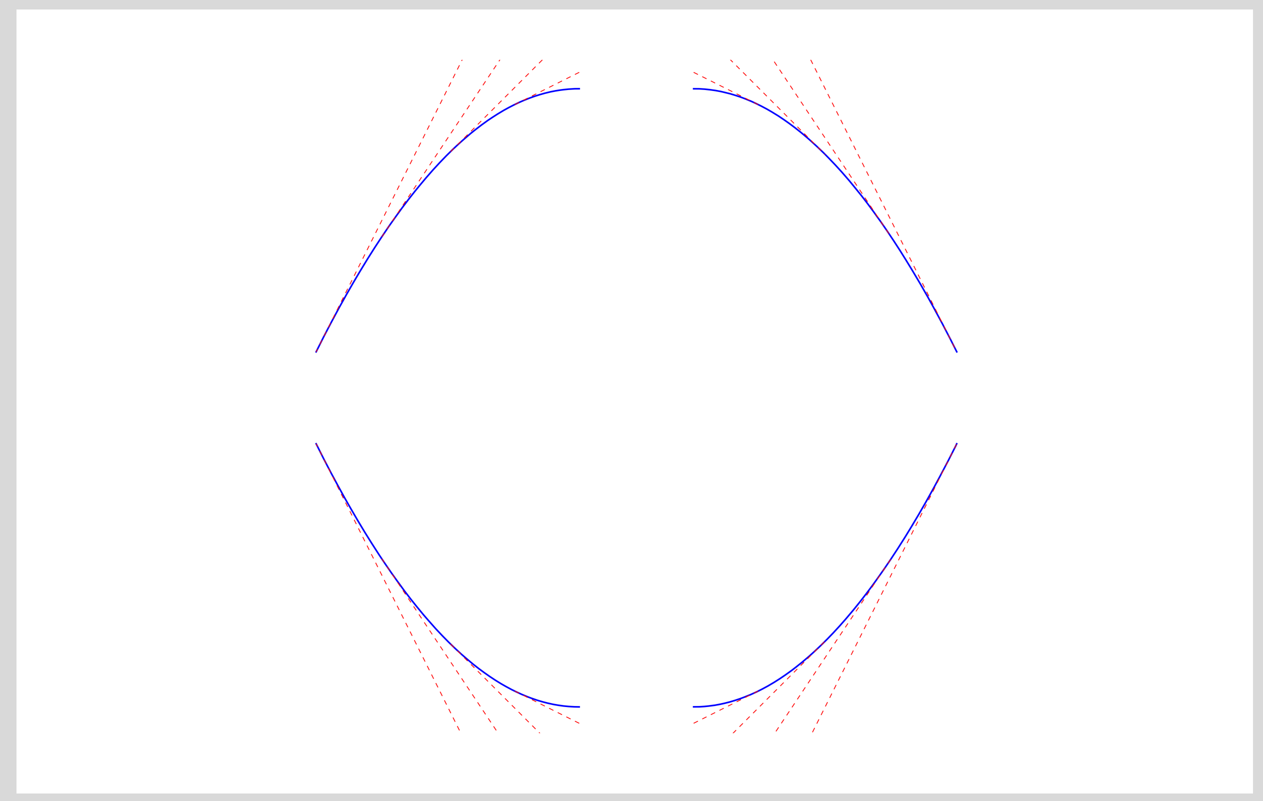
- An *increasing* curve has a minimum on the left and a maximum on the right; and
- a *decreasing* curve has a maximum on the left and a minimum on the right.

Which shape do we use to connect a minimum to a maximum, or vice versa? To answer that, we need a new concept.

The rate of change of the rate of change

Reexamine the four basic shapes that interest us. In each case, the derivative–the slope of the tangent line–is either increasing or decreasing.



What is the derivative doing in each plot?

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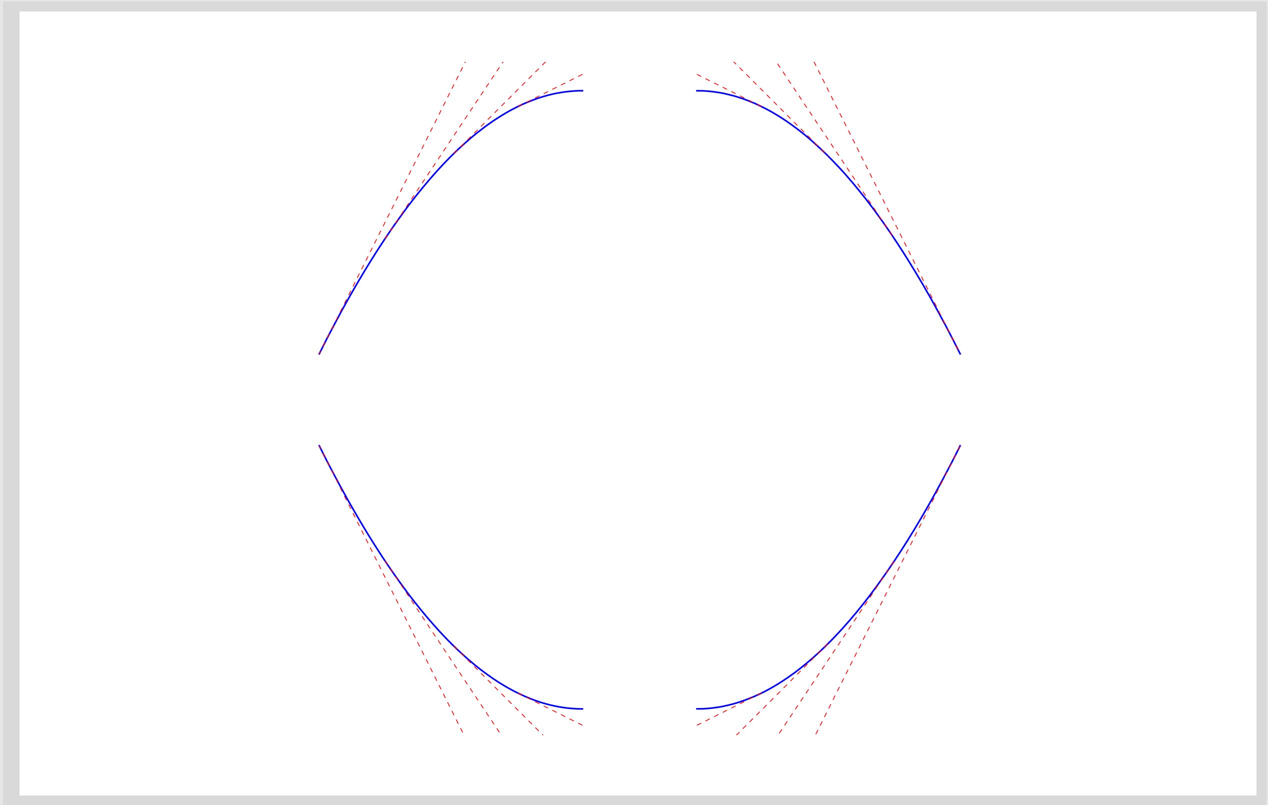
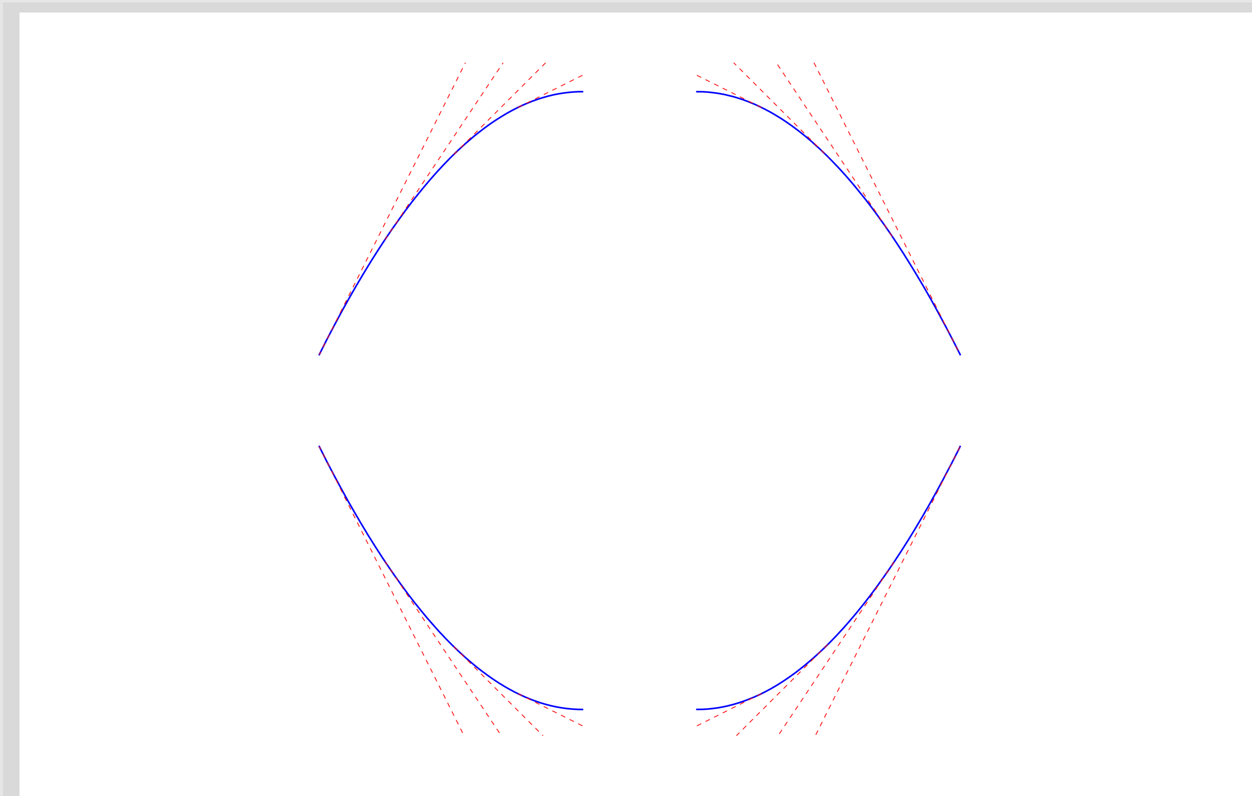


What is the derivative doing in each plot?

decreasing	decreasing
increasing	increasing

The rate of change of the rate of change

Reexamine the four basic shapes that interest us. In each case, the derivative–the slope of the tangent line–is either increasing or decreasing. We can already narrow the shape



of a curve to increasing or decreasing curves. If we can decide whether the *derivative* is increasing or decreasing, then we can decide which of the two remaining shapes to choose.

Concavity

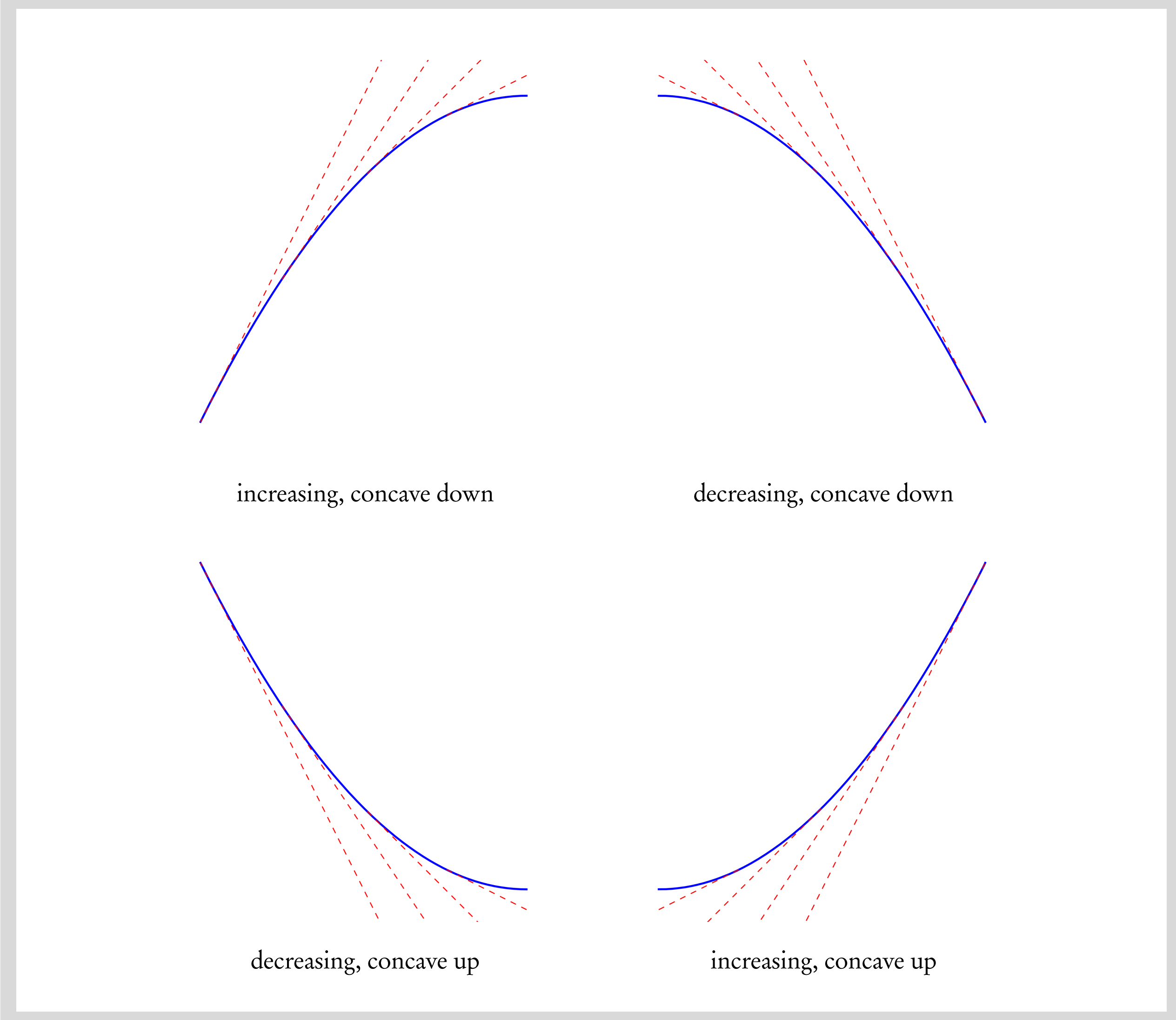
How do we decide whether the derivative is increasing or decreasing? We use the derivative of the derivative, or the *second* derivative. If f' is continuous, then the maximum and minimum values of f' tell you whether f' is increasing or decreasing. How do you find the minimum and maximum values of f'? The same way that you find the minimum and maximum values of f.

Definition: *(Concavity)*

- If f' is increasing on the interval I, then f is *concave up* on I.
- If f' is decreasing on the interval I, then f is *concave up* on I.
- if f changes concavity at $x = a$, then the point $(a, f(a))$ is called an *inflection point*.

Concavity and curve sketching

We can classify each of the four basic shapes according to the curve's (a) rate of change and (b) concavity.



Curve Sketching

We can now summarize a method for curve sketching based on finding extrema.

(Curve sketching)

- Determine asymptotes.
- Determine intervals where the curve is *increasing* or *decreasing*:
  - Find all critical points: that is, places where $f'(x) = 0$ or $f'(x)$ does not exist. These x values are called critical points.
  - Classify the critical points as local minima or maxima. This indicates the intervals where the curve is increasing or decreasing.
- Determine intervals where the curve is concave *up* or *down*:
  - Find all places where $f''(x) = 0$ or $f''(x)$ does not exist. These x values are called possible inflection points.
  - Find the values of f' at each possible inflection point *and* at the end-points.
  - Classify the possible inflection points as minima or maxima of f'. This indicates you the intervals the curve is concave up or down.
- Combine the information from the first and second derivatives to determine the shape of the curve on each interval.

First example

We apply this method to the first question posed at the beginning of this module:

Question: Make a sketch of the curve $x^3 + 2x - 1$.

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Question: Make a sketch of the curve $x^3 + 2x - 1$.

There are no asymptotes (*why not?*), so first we find the intervals where the curve is increasing or decreasing. Setting the derivative to zero, we have

$$3x^2 + 2 = 0.$$

The derivative has *no real roots*, so it is either *always increasing* or *always decreasing*.

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$$3x^2 + 2 = 0.$$

This equation has *no real roots*, so it is either *always increasing* or *always decreasing*.

Which is it? There are several ways to decide, but one easy way is to check that

at $x = 0$, $x^3 + 2x - 1 = -1$ and
at $x = 1$, $x^3 + 2x - 1 = 2$.

Since the curve must increase to rise from 0 to 1, the curve is *always increasing*.

First example

We apply this method to the first question posed at the beginning of this module:

Question: Make a sketch of the curve $x^3 + 2x - 1$.

There are no asymptotes (*why not?*). The curve is *always decreasing*. Now we must find the intervals where the curve is concave up or down.

First example

We apply this method to the first question posed at the beginning of this module:

Question: Make a sketch of the curve x^3+2x-1.

There are no asymptotes (*why not?*). The curve is *always decreasing*. Now we must find the intervals where the curve is concave up or down. Setting the second derivative to zero, we have

$$6x = 0.$$

There is a *possible* inflection point at $x = 0$.

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$$6x = 0.$$

There is a *possible* inflection point at $x = 0$.

Since no endpoints have been specified, we can check the value of the first derivative around $x = 0$ to see whether it is a maximum or a minimum of the first derivative.

at $x = -1$, $3x^2 + 2 = 5$,

at $x = 0$, $3x^2 + 2 = 2$, and

at $x = 1$, $3x^2 + 2 = 5$.

Thus $x = 0$ is a *minimum* of the first derivative.

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$$6x=0.$$

There is a *possible* inflection point at $x=0$.

Since no endpoints have been specified, we can check the value of the first derivative around $x=0$ to see whether it is a maximum or a minimum of the first derivative.

$$\text{at } x=-1, 3x^2+2=5,$$
$$\text{at } x=0, 3x^2+2=2, \text{ and}$$
$$\text{at } x=1, 3x^2+2=5.$$

Thus $x=0$ is a *minimum* of the first derivative. This tells us that

- $f'(x)$ is *decreasing* on $(-\infty,0) \Longrightarrow f(x)$ is concave *down* on $(-\infty,0)$; and
- $f'(x)$ is *increasing* on $(0,\infty) \Longrightarrow f(x)$ is concave *up* on $(0,\infty)$.

First example

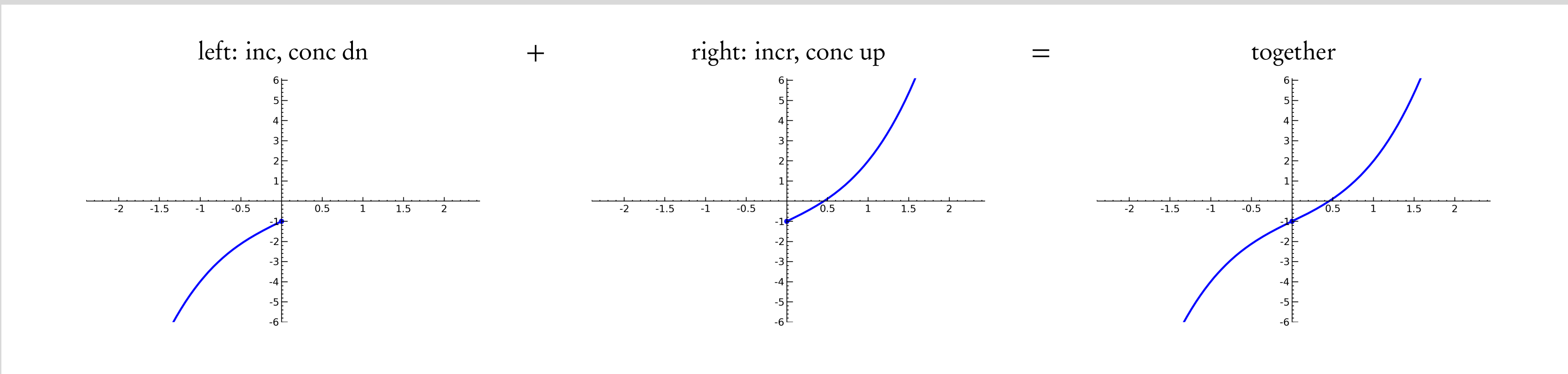
We apply this method to the first question posed at the beginning of this module:

Question: Make a sketch of the curve x^3+2x-1.

There are no asymptotes (*why not?*). The curve is *always decreasing*. It is

- concave *down* on $(-\infty, 0)$; and
- concave *up* on $(0, \infty)$.

When $x = 0$, $y = -1$. That suggests the following shape of the curve:



Second example

Now turn to the second question:

Question: Sketch the curve of $-10x^3 + \frac{x}{10} + \frac{1}{10}$, labeling every point where

- the curve starts and stops increasing to decreasing; and
- the curve's slope starts and stops increasing and decreasing.

Second example

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Question: Sketch the curve of $-10x^3 + \frac{x}{10} + \frac{1}{10}$, labeling every point where

- the curve starts and stops increasing to decreasing; and
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Again the are no asymptotes *(why not?)* so we look for critical points. Take a moment to find the extrema of the function.

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- the curve's slope starts and stops increasing and decreasing.

Again the are no asymptotes *(why not?)* so we look for critical points. Take a moment to find the extrema of the function.

Checking y values, we find these extrema:

$$\text{local max at } x = \frac{1}{10\sqrt{3}} \approx 0.058,\ y \approx 0.104 \text{ and}$$
$$\text{local min at } x = -\frac{1}{10\sqrt{3}} \approx -0.058,\ y = \approx 0.096.$$

These are very close to each other—you probably would not notice them if you merely glanced at a graph.

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$$\text{local max at } x = \frac{1}{10\sqrt{3}} \approx 0.058, \; y \approx 0.104 \text{ and}$$
$$\text{local min at } x = -\frac{1}{10\sqrt{3}} \approx -0.058, \; y = \approx 0.096.$$

That implies that

- *increases* on $(-0.058, 0.058)$, while it
- *decreases* on $(-\infty, -0.058)$ and $(0.058, \infty)$.

Second example

Now turn to the second question:

Question: Sketch the curve of $-10x^3+\frac{x}{10}+\frac{1}{10}$, labeling every point where

- the curve starts and stops increasing to decreasing; and
- the curve's slope starts and stops increasing and decreasing.

Again the are no asymptotes. The curve increases on $(-0.058, 0.058)$, while it decreases on $(-\infty, -0.058)$ and $(0.058, \infty)$. Now find the extrema of f'.

Second example

Now turn to the second question:

Question: Sketch the curve of $-10x^3 + \frac{x}{10} + \frac{1}{10}$, labeling every point where

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- the curve's slope starts and stops increasing and decreasing.

Again the are no asymptotes. The curve increases on $(-0.058, 0.058)$, while it decreases on $(-\infty, -0.058)$ and $(0.058, \infty)$. Now find the extrema of f'.

There is only one extremum of f': at $x = 0$, there is a local maximum of f'. So f' is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, which means that the f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.

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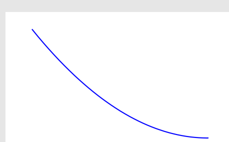
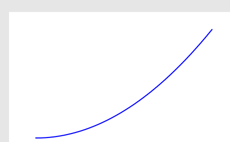
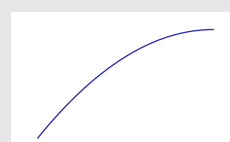
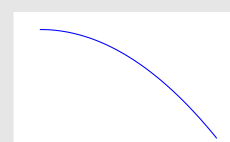
We divide the intervals as follows:

x	$(-\infty, -0.058)$	$(-0.058, 0)$	$(0, 0.058)$	$(0.058, \infty)$
incr/decr	decr	incr	incr	decr
concavity	up	up	down	down
shape				

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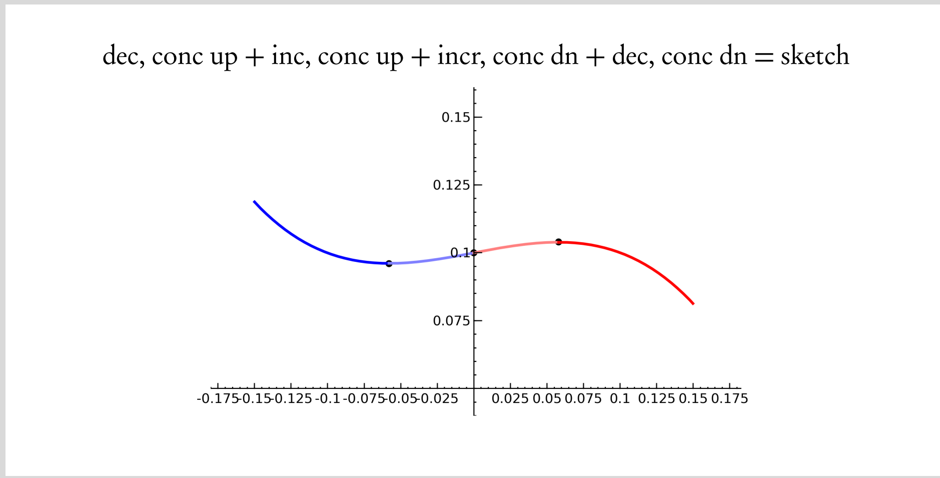
- the curve starts and stops increasing to decreasing; and
- the curve's slope starts and stops increasing and decreasing.

Again the are no asymptotes. The curve increases on $(-0.058, 0.058)$, while it decreases on $(-\infty, -0.058)$ and $(0.058, \infty)$. The curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.

Evaluating y values and assembling the pieces, we obtain:



AlternateMethod

An alternate method to determining the intervals where f is increasing, decreasing, concave up, and concave down is the following.

- Determine the critical points.
- Determine the possible points of inflection.
- Divide the real line into intervals at the critical points and possible points of inflection.
- From each interval choose a sample point. Evaluate f' and f'' at that sample point.
- If $f' > 0$ at the sample point, then f is increasing on the entire interval; otherwise f is decreasing on the entire interval.
- If $f'' > 0$ at the sample point, then f is concave up on the entire interval; otherwise f is concave down on the entire interval.

Second example: alternate method

We illustrate the alternate method using the second example. We already found the critical points and possible points of inflection, so we can divide the real line into intervals:

	$-\infty$ $\quad -0.058$	-0.058 $\quad 0$	0 $\quad 0.058$	0.058 $\quad \infty$
f'	neg	pos	pos	neg
f''	pos	pos	neg	neg

(We evaluate f' and f'' at the sample points -1, -0.01, 0.01, and 1.)

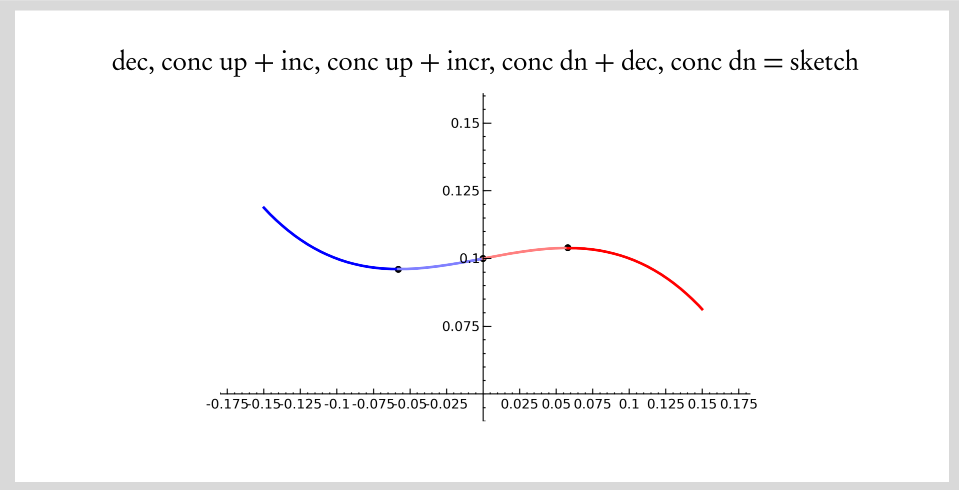
Second example: alternate method

We illustrate the alternate method using the second example. We already found the critical points and possible points of inflection, so we can divide the real line into intervals:

	$-\infty$ $\quad$ -0.058	-0.058 $\quad$ 0	0 $\quad$ 0.058	0.058 $\quad$ ∞
f'	neg	pos	pos	neg
f''	pos	pos	neg	neg

(We evaluate f' and f'' at the sample points -1, -0.01, 0.01, and 1.)

We can then assemble the pieces of the graph the same way as before:



Proof of the alternate method

This alternate method is based on the *Mean Value Theorem*. Recall that it states

Theorem: *(Mean Value Theorem)*

Suppose that a function f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Essentially, f is increasing if, and only if, $f(b)-f(a)$ is positive. If $f'(c)$ is *negative* for all $c\in(a,b)$, then the fact that $b-a$ is positive means that, by the Mean Value Theorem, $f(b)-f(a)$ must is negative.

Conclusion

- We showed how to sketch the graphs of curves of a function.
- There are two different methods to use, one based on analysis of extrema and the other based on the Mean Value Theorem. Both methods give identical results.
- Curve sketching requires an identification of *concavity*:

Definition: *(Concavity)*

- If f' is increasing on the interval I, then f is *concave up* on I.
- If f' is decreasing on the interval I, then f is *concave up* on I.
- if f changes concavity at $x = a$, then the point $(a, f(a))$ is called an *inflection point*.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the **⟨Print⟩** icon, and then saving or printing the pdf file.

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The University of
Southern Mississippi