Self-paced

# Student Study Modules

for

# Calculus I–Calculus III

## Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

## Overview

In this lesson we introduce the *fundamental* concept of Calculus: *the limit*. For now we will take a more intuitive approach, looking at *graphical* and *numerical* approaches to the limit. We will introduce a rigorous, *algebraic* approach to the limit in another module.



## Two sample problems

Let

- $f(x) = \dfrac{x^2 - x}{x}$, and
- $w(x) = \sin\left(\dfrac{1}{x}\right)$.

Both of these functions are undefined *at* $x = 0$. We want to explore how they behave as $x$ *approaches* $0$.

# The limit

**Definition:** *(Intuitive definition of a limit)*
The limit indicates the value to which a variable or a function grows arbitrarily close. We write $\lim_{x\to a} f(x) = L$ to mean that the $y$ values of $f(x)$ grow arbitrarily close to $L$ when the $x$ values grow arbitrarily close to $a$.

In the case examples listed on the previous page, we want to know the *limit* of $f$ or $w$ as $x$ approaches 0.

In general, when we want to know the *limit* of $f$ as $x$ approaches some value $a$, we write

$$\lim_{x\to a} f(x).$$

In this worksheet, we want to evaluate

$$\lim_{x\to 0} f(x) \quad \text{and} \quad \lim_{x\to 0} w(x).$$



# SAGE worksheets

In this lab you will not need a special SAGE worksheet, but you will need to have a blank SAGE worksheet handy, so go ahead and login to SAGE and open a new worksheet.



# The numerical approach

The first method to evaluating a limit is called the *numerical approach*. To estimate the limit of $f$ as $x$ approaches 0, we build a table of $x$-$y$ values using values of $x$ that get close to 0. We should choose values that lie on both sides of 0, in order to make sure the limits are the same from left *and* right.

First we build the table:

| $x$ | -1 | -0.1 | -0.01 | 0 | 0.01 | 0.1 | 1 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | ??? | | | |

(Recall that $y = f(x) = \frac{x^2 - x}{x}$.)




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Now we fill in the table:

| $x$ | -1 | -0.1 | -0.01 | 0 | 0.01 | 0.1 | 1 |
|---|---|---|---|---|---|---|---|
| $y$ | -2 | -1.1 | -1.01 | ??? | -0.99 | -0.9 | 0 |

Can you guess the value of $\lim_{x\to 0} f(x)$?

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| $y$ | -2 | -1.1 | -1.01 | -1 | -0.99 | -0.9 | 0 |

It *looks* as if the $y$-values are approaching 0. This suggests that

$$\lim_{x\to 0} f(x) = -1.$$



# SAGE and numerical computations

You have several options for creating the table below.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | -1 | -0.1 | -0.01 | 0 | 0.01 | 0.1 | 1 |
| $y$ | -2 | -1.1 | -1.01 | ??? | -0.99 | -0.9 | 0 |

- You can find these values by hand. This function isn't very difficult, so that's easy.
- You can use a calculator. This can be rather tiresome on the fingers when the function is involved, but it probably saves times from hand computation. (If your calculator has a table feature, your fingers won't be so tired.)
- You can use SAGE. How?

Assignment: the intuitive notion of a limit 5

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  - The simplest way is to define the function $f$ by typing `f = (x^2-x)/x`. To evaluate it at $x=-1$, type `f(-1)`. Repeat this for other values.

Assignment: the intuitive notion of a limit 5

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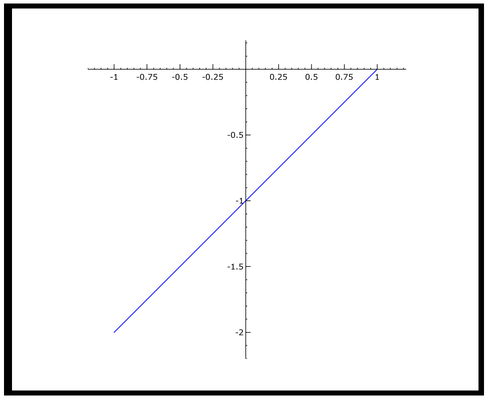
- You can use SAGE. How?
  - The simplest way is to define the function $f$ by typing `f = (x^2-x)/x`. To evaluate it at $x=-1$, type `f(-1)`. Repeat this for other values.
  - A somewhat faster method is to use a loop. You can do this by typing `[f(a) for a in [-1,-0.1,-0.01,0.01,0.1,1]]`. (*Try it!*) In this `for` loop, the variable `a` takes all the values specified in the list, and the expression `f(a)` is evaluated for each of those values.

## The graphical approach

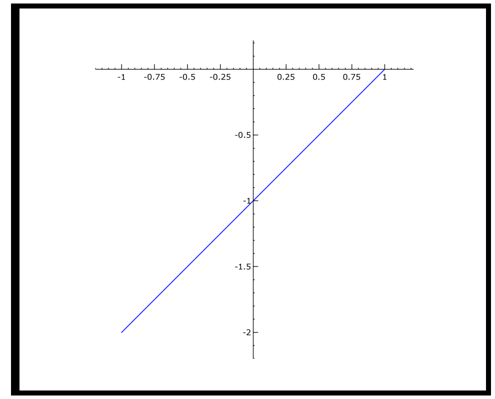
Now let's apply the graphical approach to $f(x)$. Sketch a graph of $f$ around $x=0$. You should see something like this:






## The graphical approach: WARNING

Now let's apply the graphical approach to $f(x)$. Sketch a graph of $f$ around $x=0$. You should see something like this:

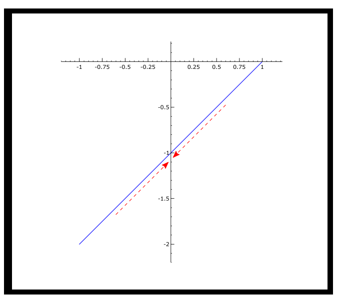



You have to be careful reading plots that are generated by a computer. The computer doesn't know how to discover the division by zero, so it draws the curve as if there were a point at $(0,-1)$ *even though there is no point there*.



## The graphical approach: the limit

Now let's apply the graphical approach to $f(x)$. Sketch a graph of $f$ around $x=0$. You should see something like this:



It is often easy to "see" the limit from a graph. Recall that $\lim_{x\to 0} f(x)$ is the value that $f$ approaches as $x$ approaches $0$. Imagine that you are an ant walking along the graph of $f$. What $y$ value would you approach when your $x$ value approaches $0$? That point is the *limit* of $f$ as $x$ approaches $0$. Here

$$\lim_{x\to 0} f(x) = -1.$$



## SAGE and plots

As with the numerical approach, you have several options for sketching a graph.

- You can sketch it by hand using an $x$-$y$ table. This takes a long time, and is subject to human error.
- You can use a graphing calculator. This doesn't take very long, but requires a $100 calculator that isn't very flexible in its graphing.
- You can use SAGE's `plot()` function. This doesn't take very long, does not require any money besides what you've already paid for the class, and is very flexible. See the module "An Introduction to SAGE" for more details.

## Specificity of a limit

Notice that we have talked about computing or looking from *both* sides. The limit can have only *one specific* value. It should be the same whether $x$ approaches $a$ from negative values ("from the left") or whether $x$ approaches $a$ from positive values ("from the right"). We write this property as

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

where $x \to a^-$ reads "as $x$ approaches $a$ from the left" and $x \to a^+$ reads "as $x$ approaches $a$ from the right".



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$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

Both the table of $x$-$y$ values and the graph suggested that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x),$$

so we were able to conclude that $\lim_{x\to 0} f(x) = 0$; that is, the *two-sided limit* exists.



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$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

On the other hand, the function $g$ graphed below has *no* limit at $x = 0$, because the limits from the left and right disagree.

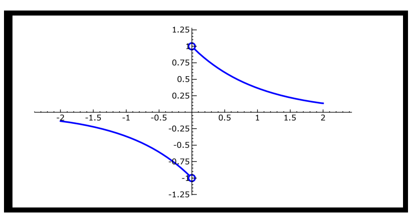


Figure 1: $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$



## Second example

Now let's try to estimate

$$\lim_{x\to 0} w(x)$$

where

$$w(x) = \sin\left(\frac{1}{x}\right).$$

## Second example: The numerical approach

To estimate the limit of $w$ as $x$ approaches $0$, we build a table of $x$-$y$ values using values of $x$ that get close to $0$.

Fill in the table, rounding to the ten-thousandths place.

| $x$ | -1.0 | -0.1 | -0.01 | 0 | 0.01 | 0.1 | 1.0 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | ??? | | | |



## Second example: The numerical approach

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Fill in the table, rounding to the ten-thousandths place.

| $x$ | -1.0 | -0.1 | -0.01 | 0 | 0.01 | 0.1 | 1.0 |
|---|---|---|---|---|---|---|---|
| $y$ | -0.8415 | 0.5540 | 0.5063 | ??? | -0.5063 | -0.5440 | 0.8415 |

Can you guess the value of $\lim_{x\to 0} w(x)$?



## Second example: The numerical approach

To estimate the limit of $w$ as $x$ approaches $0$, we build a table of $x$-$y$ values using values of $x$ that get close to $0$.

Fill in the table, rounding to the ten-thousandths place.

| $x$ | -1.0 | -0.1 | -0.01 | 0 | 0.01 | 0.1 | 1.0 |
|---|---|---|---|---|---|---|---|
| $y$ | -0.8415 | -0.5540 | -0.5063 | 0.5? | 0.5063 | 0.5440 | 0.8415 |

You might try to "guess" from this data that

$$\lim_{x\to 0} w(x) \approx 0.5.$$

You might also try to "guess" that it should be the average; that is,

$$\lim_{x\to 0} w(x) = 0.$$

*That would be a very bad guess.*



## Second example: The graphical approach

The graphical approach will show us why we cannot say that the limit is either of those values. Sketch the graph of $w(x)$.

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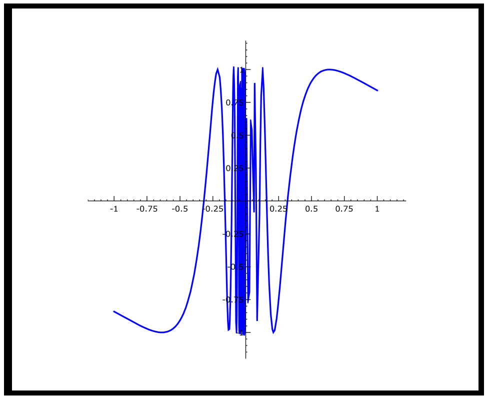
You should see the following:

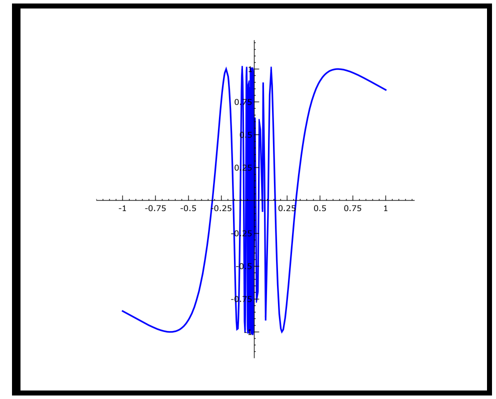



What is going on?



## The infinite wiggle

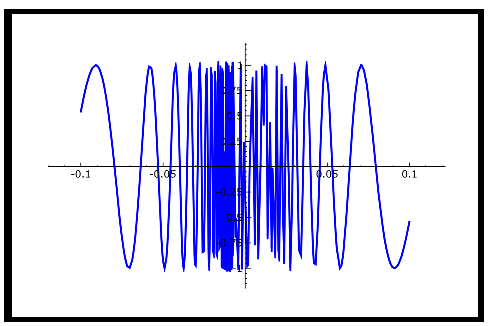
This graph is something called an *infinite wiggle*. As the curve gets closer to $x = 0$, the $y$-values start to vibrate more and more wildly between $y = -1$ and $y = 1$.






## The infinite wiggle zoomed

This graph is something called an *infinite wiggle*. As the curve gets closer to $x = 0$, the $y$-values start to vibrate more and more wildly between $y = -1$ and $y = 1$.




You can zoom in closer if you wish; $w$ wiggles all the more. It's a regular party down there.



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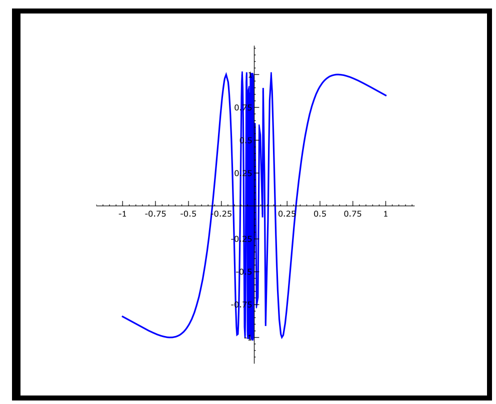


What then should we say about the limit of $w$ as $x$ approaches $0$? The function does not approach *one* specific value, so we say that the limit *does not exist*.

## Fixing the numerical approach

We can see this clearly by computing more $x$-$y$ values in the numerical approach. Here is a slightly larger table of values approaching from the left:

| $x$ | -1.0 | -0.1 | -0.01 | -0.001 | -0.0001 | -0.00001 |
|---|---|---|---|---|---|---|
| $y$ | 0.8415 | 0.5540 | 0.5064 | $-0.8269$ | 0.3056 | $-0.0357$ |

The numbers they bounce around, and do not appear to approach any one value. This also provides a clue that the function is not settling on a limit.



## Use both methods carefully

To estimate a limit, it is probably a good idea to use both methods. You should build $x$-$y$ tables with a *lot* of carefully chosen $x$-values, and you should zoom into your graphs to make sure that no strange behaviors are disguised by too large a viewing area.

- Sometimes numerical values can appear to bounce around, but a graph will show that there is a trend for the bouncing to even out as the $y$-values approach a definite number. For an example of this phenomenon, look at a plot of the function $x(1+\sin(100x))$ on the interval $[-1,1]$.
- On the other hand, the graph of a function may sometimes suggest a limit when none exists, whereas a numerical approach will show that the values are in fact bouncing around. For an example of this phenomenon, look at a plot of the function $0.01\sin(1/x)+x$ on the interval $[-1,1]$.



## Estimation vs. evaluation

All the same, these methods only allow us to *estimate* a limit, they do not allow us to *evaluate* the limit. We will develop a precise method to help us evaluate limits *precisely* in a future module.



## Conclusion

- The *limit* of a function $f$ as $x$ approaches a value $a$ is the value $f$ approaches as we close in on $x=a$ from the left and right sides of $a$.
- There can only be one value for a limit. Limits do not always exist.
- We studied two methods of *estimating* limits:
  - a *numerical* method, and
  - a *graphical* method.
- Using both these methods can give us an idea of what the value of a limit is, but they do not give us *certainty* as to what the value of a limit is.

# End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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