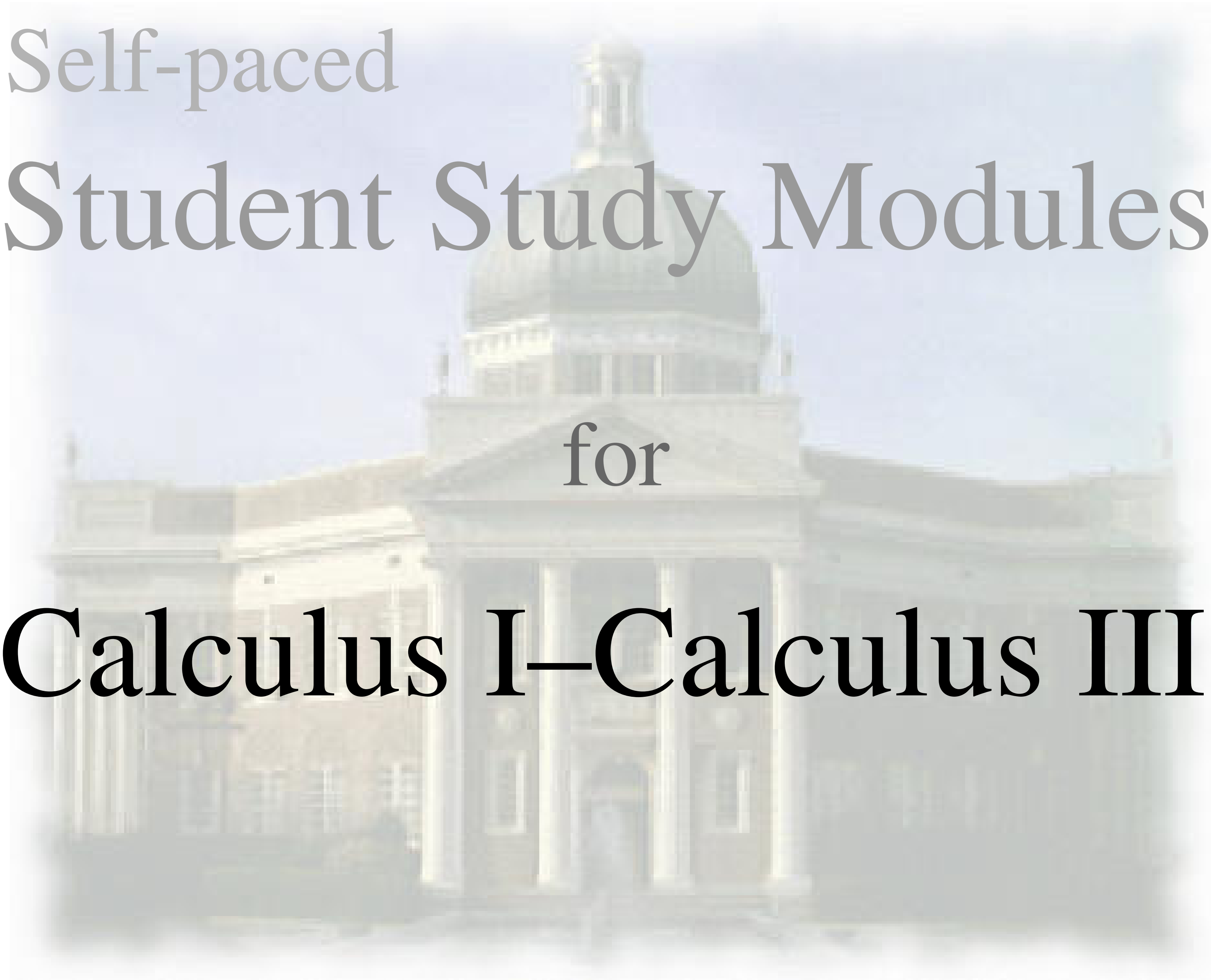

Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages **⟨Next⟩**, backward **⟨Prev⟩**, or view all the slides in this tutorial **⟨Index⟩**.
- The **⟨Back to Calc I⟩** button returns you to the course home page.
- A full symbolic algebra package **⟨Sage⟩** is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text **⟨CalcText⟩** provides a quick search of basic calculus topics.
- You can get help from Google Calculus **⟨GoogleCalc⟩**.
- A monochrome copy of this module is suitable for printing **⟨Print⟩**.

When all else fails, feel free to contact your instructor.

Extrema

A common problem is one of find extrema, which are minimum and maximum values of a function. Methods of approximating an exremum are possible, but generally unsatisfactory. Derivatives give us an way to compute extrema in a very efficienty manner.

Sample problems

Question:

- Find the maximum and minimum values of xe^x on the interval $[-10,10]$.
- Where does the graph of a function change direction from increasing to decreasing or vice versa?
- A manufacturer's profit depends on the volume of sales: if not enough units are sold, income is insufficient to cover the costs, but if too many units are sold, the costs of making too much of a product grow faster than income. The profit function for the manufacturer Widgets-R-Us is known to be $p(x) = -x^3 + 2.66x^2 + 3.8649x + 0.113526$, where x is the number of units sold, in thousands. What number of units sold should the manufacturer target in order to maximize profit?

SAGE worksheets

For this module you will need a blank SAGE worksheet.

Extrema: Definition

An extremum is either a minimum or a maximum value of a function.

- The function $f(x)$ has a maximum at $x = c$ on the interval $[a, b]$ if $f(c) > f(x)$ for all other values $x \in [a, b]$.
- The function $f(x)$ has a minimum at $x = c$ on the interval $[a, b]$ if $f(c) < f(x)$ for all other values $x \in [a, b]$.

Extrema: Approximation

The first example asks us to

> **Question:** Find the maximum and minimum values of xe^x on the interval $[-10, 10]$.

We can approximate the minimum by looking at y values of xe^x on the interval. Use SAGE to make a table of values and approximate the extrema.

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(*Hint:* The sequence of commands

```
f = x*e^x
[round(f(each),2) for each in [-10,-9,-8,-7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10]]
```

will compute the values of $f(-10)$, $f(-9)$, $\ldots$, $f(10)$ rounded to the hundredths place.)

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(*Hint:* Another trick that works is the sequence of commands

```
f = x*e^x
[round(f(each-10),2) for each in range(21)]
```

because the command `range(21)` gives a list of numbers from 0 to 20.)

Extrema: Approximation

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Checking every integer in the interval, we see the values

x	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
xe^x	-0.0	-0.0	-0.0	-0.01	-0.01	-0.03	-0.07	-0.15	-0.27	-0.37	0.0

x	1	2	3	4	5	6	7
xe^x	2.72	14.78	60.26	218.39	742.07	2420.57	7676.43

x	8	9	10
xe^x	23847.66	72927.76	220264.66

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x	8	9	10
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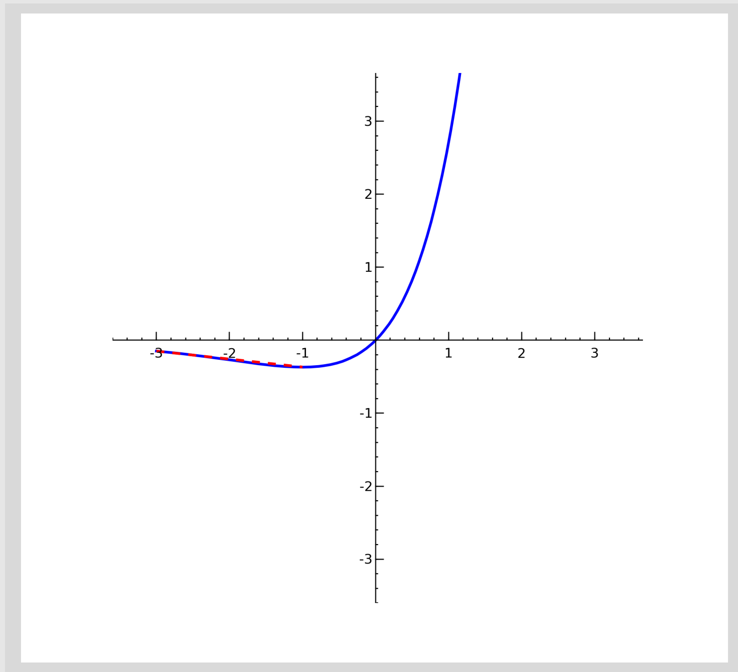
It *looks* as if the minimum and maximum values are approximately -0.37 and 220264.66. They appear at $x = -1$ and $x = 10$. How can we be *certain* about this, and how can we find the *precise* minimum and maximum?

Extrema: Certainty of computation

Think about what must happen to the y values around a minimum $m = f(c)$ at $x = a$.



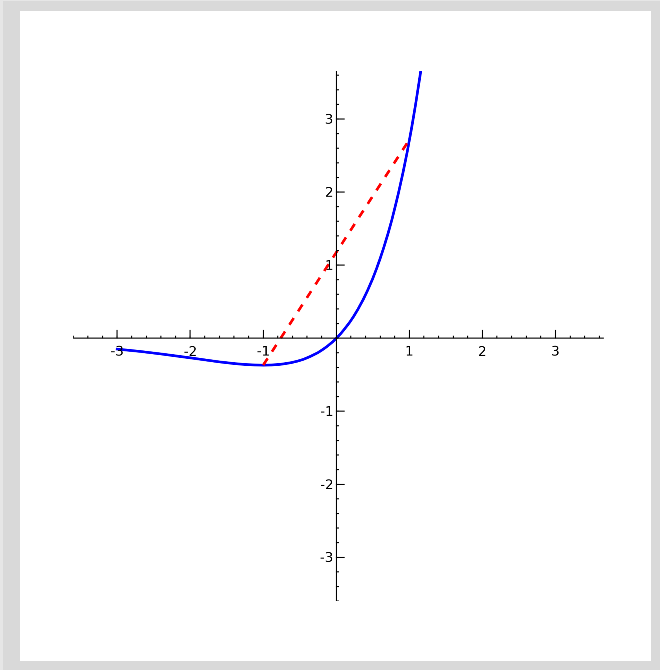
- The y values should be larger than m, because m is a minimum.
- If $x < c$ but close to c, the slopes of the secant lines will be negative:

$$\frac{f(x) - f(c)}{x - c} = \frac{\text{positive}}{\text{negative}}$$

so the slopes of the tangent lines should also be negative. Thus the derivative should be negative for $x < c$.

Extrema: Certainty of computation

Think about what must happen to the y values around a minimum $m = f(c)$ at $x = a$.



- The y values should be larger than m, because m is a minimum.
- If $x > c$ but close to c, the slopes of the secant lines will be positive:

$$\frac{f(x) - f(c)}{x - c} = \frac{\text{positive}}{\text{positive}}$$

  so the slopes of the tangent lines should also be positive. Thus the derivative should be positive for $x > c$.

Extrema: Certainty of computation

The preceding argument implies the following result.

> **Theorem:** If the function $f(x)$ is continuous on the interval $[a, b]$, it has a minimum at $x = c$ when
>
> - $f'(x) < 0$ for all $x \in [a, c)$, and
> - $f'(x) > 0$ for all $x \in (c, b]$.

A similar theorem exists for a maximum, but reversing the inequalities:

- $f'(x) > 0$ for all $x \in [a, c)$, and
- $f'(x) < 0$ for all $x \in (c, b]$.

How to find extrema

If $f(x)$ is continuous on the interval $[a,b]$, you can find extrema in the following way:

- Find all places where $f'(x)=0$ or $f'(x)$ does not exist. These x values are called critical points.
- Find the y values at each critical points *and* at the endpoints.
- The smallest y value found is the minimum, and the largest y value found is the maximum.

First example

Recall the first example:

> **Question:** Find the maximum and minimum values of xe^x on the interval $[-10, 10]$.

How would we find the extrema?

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First we check where $f'(x)$ is zero or undefined:

$$f'(x) = xe^x + e^x = (x+1)e^x.$$

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$$f'(x) = xe^x + e^x = (x+1)e^x.$$

This derivative is never undefined, and is zero when $x = -1$:

$$(x+1)e^x = 0 \text{ when } x+1 = 0 \text{ or } e^x = 0.$$

(Remember that e^x is never zero, so $x+1=0$ is the only equation that can give us a solution. Thus $x = -1$.)

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Now find the y values at the critical point and at the endpoints:

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Now find the y values at the critical point and at the endpoints:

x	-10	-1	10
$f(x)$	$-10e^{-10} \approx 0.00$	$-e^{-1} \approx -0.37$	$10e^{10} \approx 220264.66$

The minimum is $-e^{-1} \approx -0.37$ at $x = -1$. The maximum is $10e^{10} \approx 220264.66$ at $x = 10$.

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Now we turn to the second example:

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> **Question:** Where does the graph of a function change direction from increasing to decreasing or vice versa?

The graph of a function *increases* when, if the x values are increasing, so are the y values. The graph of a function *decreases* when, if the x values are increasing, the y values are decreasing. This is the same as finding the extrema. The graph of a function changes direction from increasing to decreasing at any point where the derivative is zero, or where it is undefined.

Third Example

The third example was a word problem.

> **Question:** A manufacturer's profit depends on the volume of sales: if not enough units are sold, income is insufficient to cover the costs, but if too many units are sold, the costs of making too much of a product grow faster than income. The profit function for the manufacturer Widgets-R-Us is known to be $p(x) = -x^3 + 2.66x^2 + 3.8649x + 0.113526$, where x is the number of units sold, in thousands. What number of units sold should the manufacturer target in order to maximize profit?

Essentially, we want to find the maximum value of $p(x)$ on the interval $[0, \infty)$.

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Essentially, we want to find the maximum value of $p(x)$ on the interval $[0, \infty)$. The first thing to do is to find the critical points. First, compute the derivative, and find where it is zero or undefined.

$$0 = p'(x) = -3x^2 + 5.32x + 3.8649x \ = \frac{-5.32 \pm \sqrt{5.32^2 - 4 \cdot (-3) \cdot 3.8649}}{2 \cdot (-3)} \approx -0.55, 2.33.$$

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Essentially, we want to find the maximum value of $p(x)$ on the interval $[0, \infty)$.

- There is a critical point at $x = 2.33$.

The negative critical point -0.55 is outside the domain $[0, \infty)$. Since $p'(x)$ is never undefined, there are no other critical points.

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Essentially, we want to find the maximum value of $p(x)$ on the interval $[0, \infty)$.

- There is a critical point at $x = 2.33$.

Comparing the y values of the critical point and the endpoint $x = 0$, we see

$$p(2.33) = 10.91028$$
$$p(0) = 0.113526.$$

The manufacturer should sell 2,300 units to maximize profit.

Local Extrema

One can distinguish in Calculus between global extrema and local extrema. An extremum is global if it the largest or smallest y value on the entire interval specified. An extremum is local if it is the largest or smallest y value on some interval, but not perhaps the one specified.

Local Extrema: Example

As an example, we compute the local extrema of $f(x) = x \sin x$ on the interval $[0, 2\pi]$.

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The derivative is $f'(x) = \sin x + x \cos x$. To find where this is zero, you will have to use some method of approximating the zero of a function. You can do this with the "Newton's Method" SAGElet.

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As an example, we compute the local extrema of $f(x) = x\sin x$ on the interval $[0, 2\pi]$.

- You should find that critical points occur at $x = 0$, $x \approx 2.03$, and $x \approx 4.91$.

Compute the y values of the critical points and the endpoints:

x	0	2.03	4.91	2π
$f(x)$	0	1.82	-4.81	0

- At $x = 0$, we have a *local* minimum;
- at $x = 2.03$, we have a *global* maximum;
- at $x = 4.91$, we have a *global* minimum; and
- at $x = 2\pi$, we have a *global* minimum.

Local Extrema: Caution

Be careful when analyzing this, because some functions have critical points that are neither minima nor maxima! For a minimum, the function *must* decrease to the value, then increase. For a maximum, the function *must* increase the to value, then decrease.

For example, if $f(x) = x^3$, there is a critical point at $x = 0$ (check this yourself!) but it cannot be a local minimum or maximum on the interval $[-1, 1]$:

- $f'(x) > 0$ for all $x \in [-1, 0)$; and
- $f'(x) > 0$ for all $x \in (0, 1]$.

This does not meet the definition of either a minimum or a maximum, no matter how small you make the interval around $x = 0$.

Conclusion

In this module, we discussed:

- How to find the extrema, or minima of a function f:
  - find all x values where the derivative of f is zero or undefined;
  - compute the y values of f at those points and at the endpoints;
  - select the largest y value as the maximum, and the smallest y value as the minimum.
- How to find the extrema by guess and check, but that would be unreliable.
- How extrema are related to where the graph of a curve changes direction.
- How extrema can be used in applications of mathematics.

Later we will use the example of where a graph changes direction to help draw the sketches of curves.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the **⟨Print⟩** icon, and then saving or printing the pdf file.

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