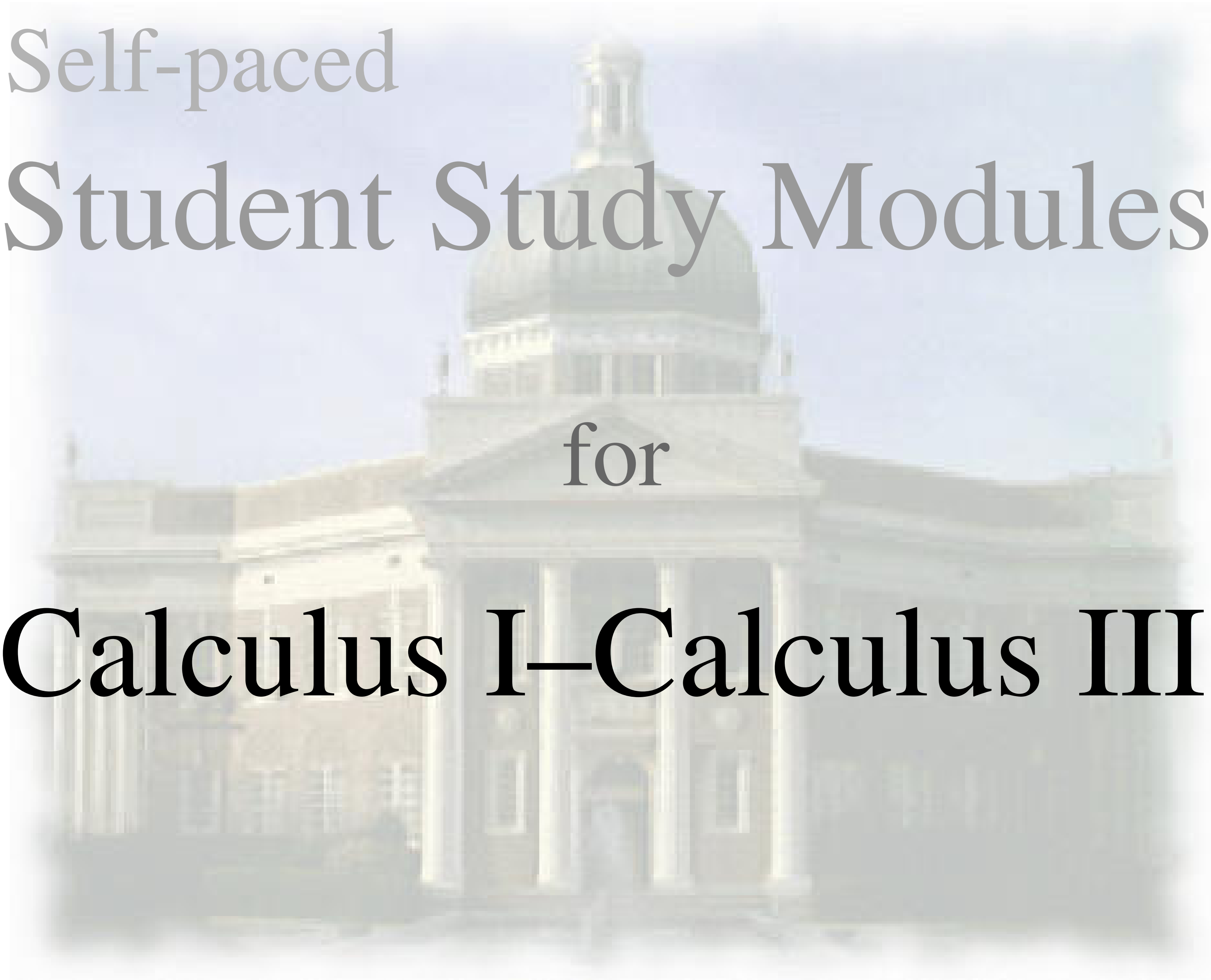


Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Newton's Method

So far we have explored two different methods for finding the root of a function: one based on the Intermediate Value Theorem, and one based on Linear Approximation. Both of those approaches are quite inefficient. In this module, we study a method developed by Isaac Newton that is much more efficient.

Sample problem

Question:

Approximate a positive zero of $x^3 + 0.4x^2 - 0.51x - 0.16$ to the ten thousandths place.

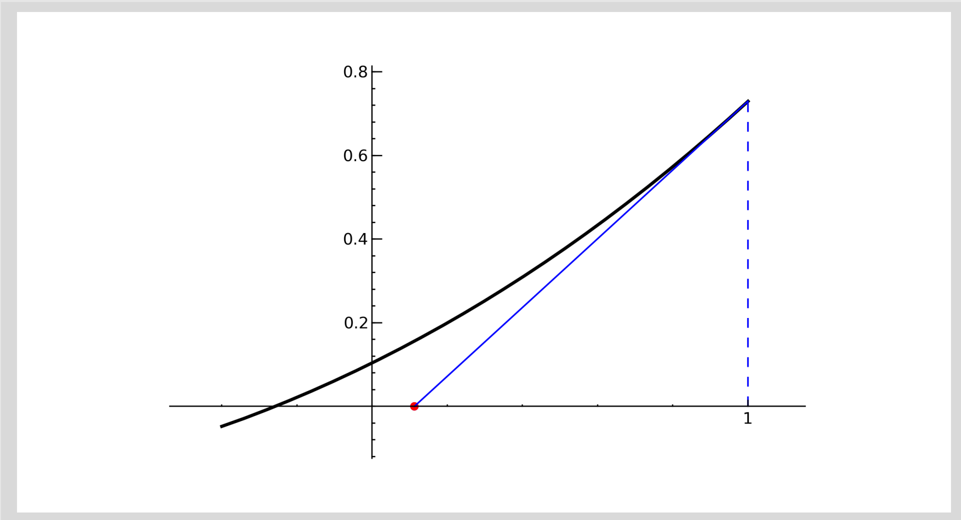
SAGE worksheets

For this module you will need the SAGE worksheet Newton's Method.

Newton's Method: the idea

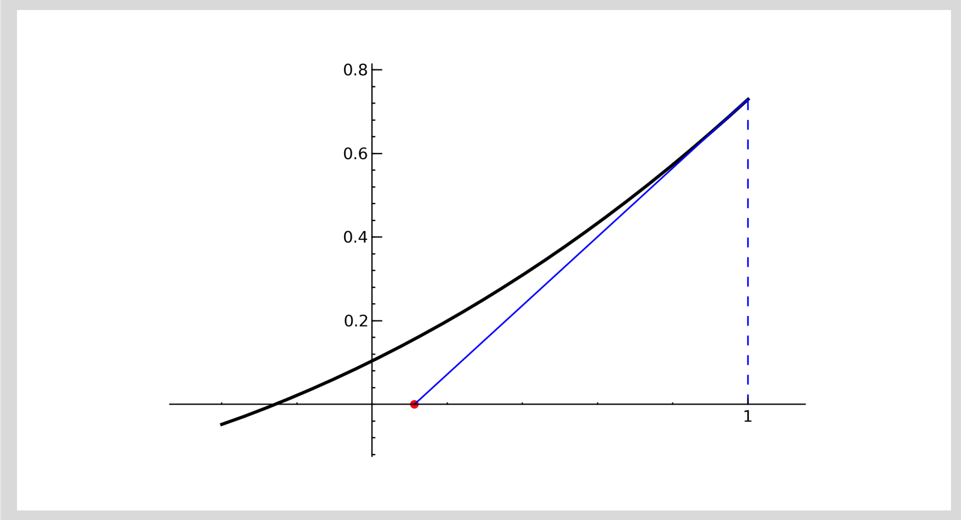
Newton's Method is based on the fact that the line tangent to a function follows the same general direction as the function. If you start "close enough" to a root of the function, the root of the tangent line will be close to the root of the function.

The illustration below gives an example. The root of f (the black curve) is approximately 0.70. A line tangent to the graph of f at $x = 1$ has a root at approximately 0.78.



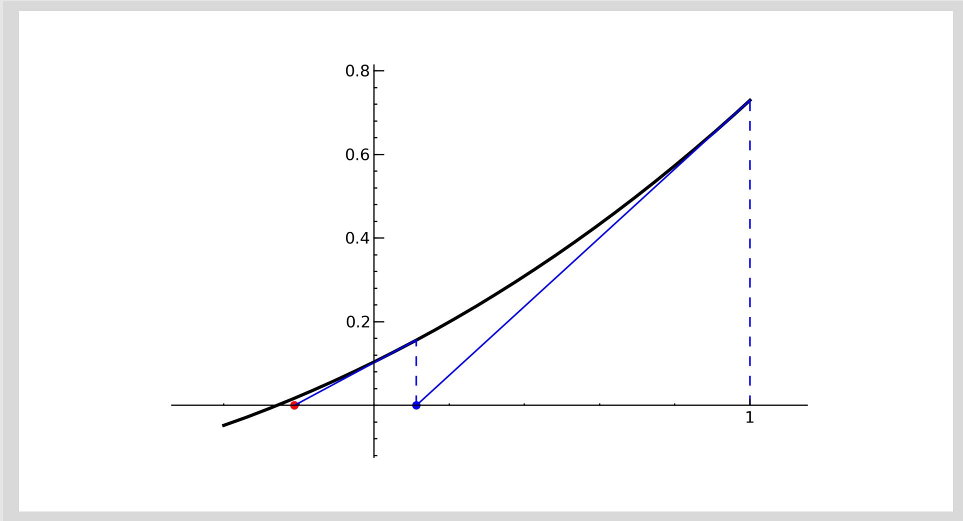
Newton's Method: the idea

By itself, 0.78 is not that good an approximation. Estimating from the graph, it looks as if $f(0.78) \approx 0.15$, which is not very close to zero. Let's build a new line tangent to the graph of f...



Newton's Method: the idea

By itself, 0.78 is not that good an approximation. Estimating from the graph, it looks as if $f(0.78) \approx 0.15$, which is not very close to zero. Let's build a new line tangent to the graph of f... at $x = 0.78$.



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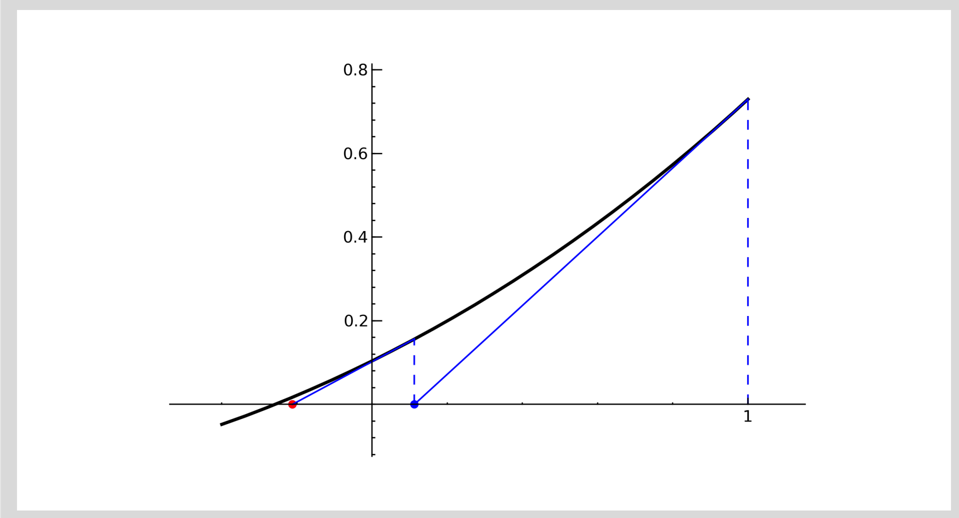


The root of the second line is approximately 0.70. The graph shows that it is closer to the zero of f.

Newton's Method: the algorithm

This experiment suggests an algorithm to find zeroes of a function.

Given f and a point a that you think is "close" to a root of f, you can approximate a root of f by repeating these steps:

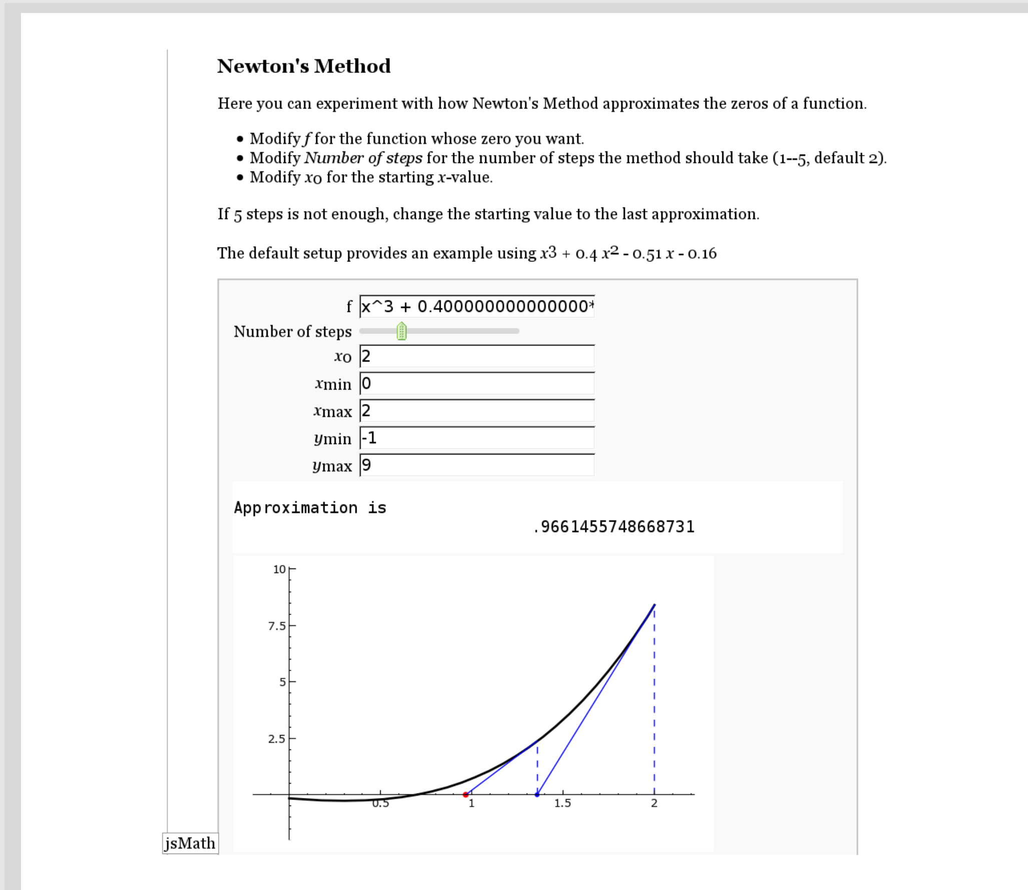
- Compute the line tangent to f at $x = a$.
- Find the root of this equation, and call it b.
- Is b "close enough" to the actual root?
  - If not, repeat the process using b in place of a.
  - If so, you are done!

"Close enough"?!?

How do we decide that our approximation is "close enough"? We will base our decision on whether the decimal digits of the current approximation (b) are the same as those of the previous approximation (a) up to the desired accuracy.

Newton's Method: example

Let's try an example. Fire up the Newton's Method SAGElet. It should start automatically, showing an approximation of the root of $x^3+4x^2-.51x-.16$ using an initial guess of $a=2$, and two tangent lines.



Newton's Method: example

The approximation given (0.9661) is clearly insufficient. Move the slider labeled "Number of approximations" to the midway point, and the applet will try three approximations. You should obtain 0.7629.

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Newton's Method: example

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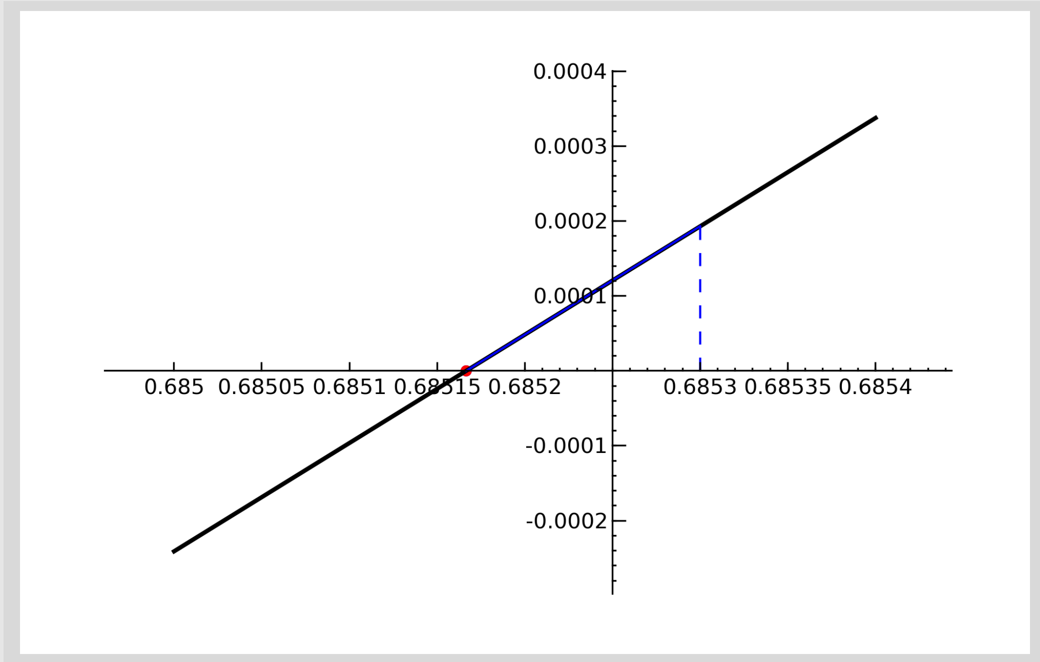
Four approximations gives 0.6937, and five gives 0.6853. We've managed to get the *first* digit to repeat, but not the rest. To obtain more accuracy, we will have to change the starting point. Move the slider all the way to the left, then change the starting point x_0 to our last approximation, 0.6853. It will be hard to see what is going on, so change the window (`xmin`, `xmax`) the details grow clear.

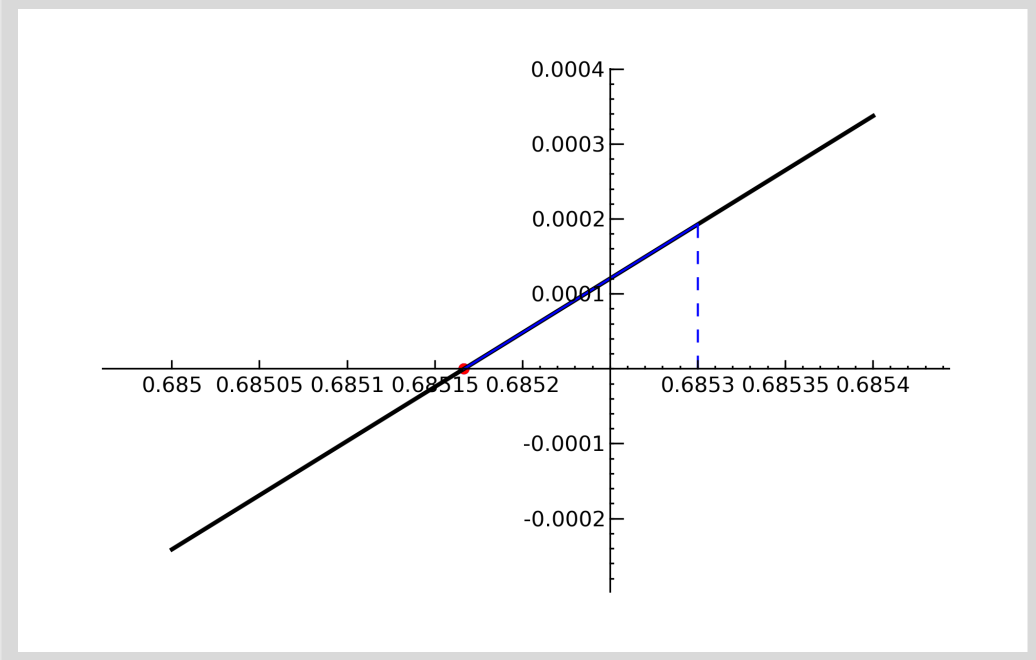
Newton's Method: example

Putting `xmin` $=$ 0.685 and `xmax` $=$ 0.6854, we obtained the following diagram.



Newton's Method: example

Putting `xmin` = 0.685 and `xmax` = 0.6854 we obtained the following diagram.



The applet reports that running Newton's method one step from 0.6853 gives the approximation 0.6852. We've now managed to keep *three* digits the same.

Newton's Method: exampe

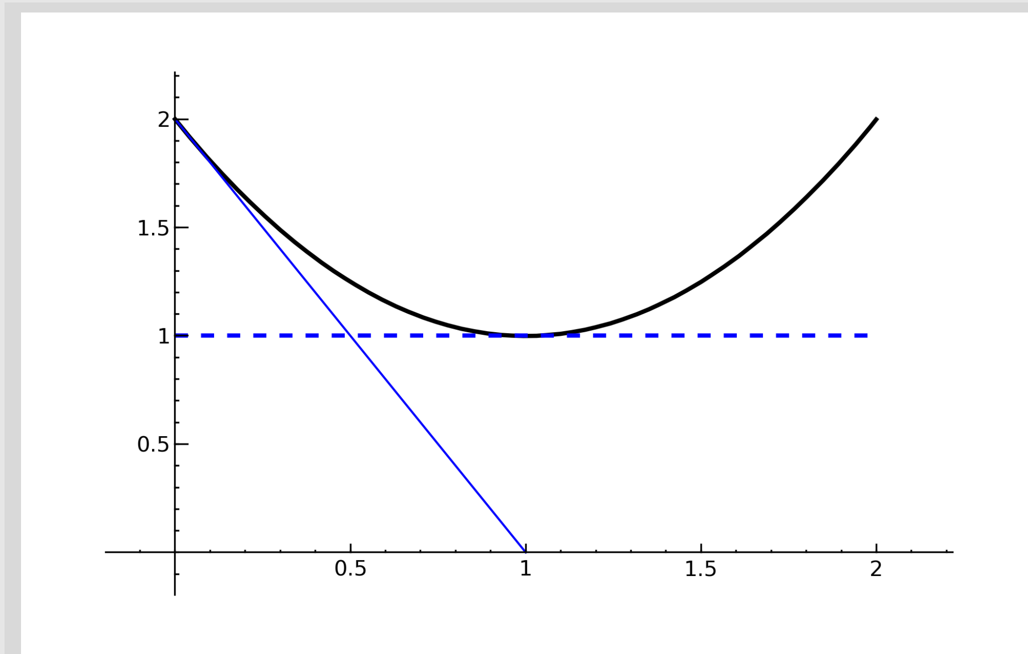
Move the slider right one spot, and you should obtain the new approximation 0.6852. We've managed to keep four digits the same, so we will decide that this approximation is *close enough*, or at least accurate to the ten thousandths place.

(In fact, we've managed to keep six digits the same.)

Newton's Method: Dangers

There are several cautions to observe with Newton's Method.

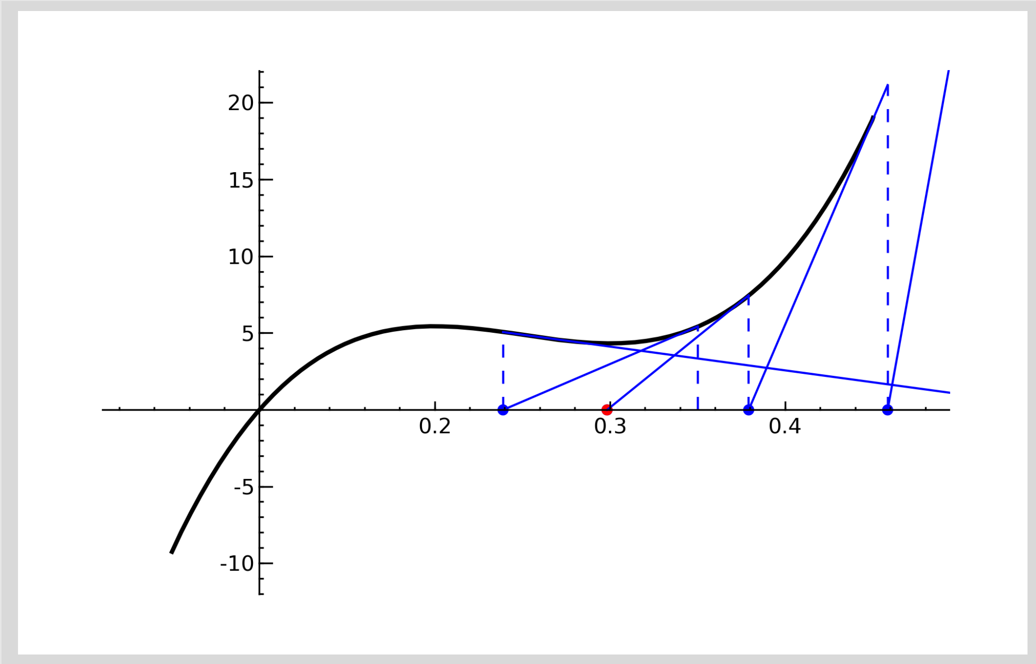
- If we encounter a point whose tangent line is horizontal, the method fails because the line has no root.



Newton's Method: Dangers

There are several cautions to observe with Newton's Method.

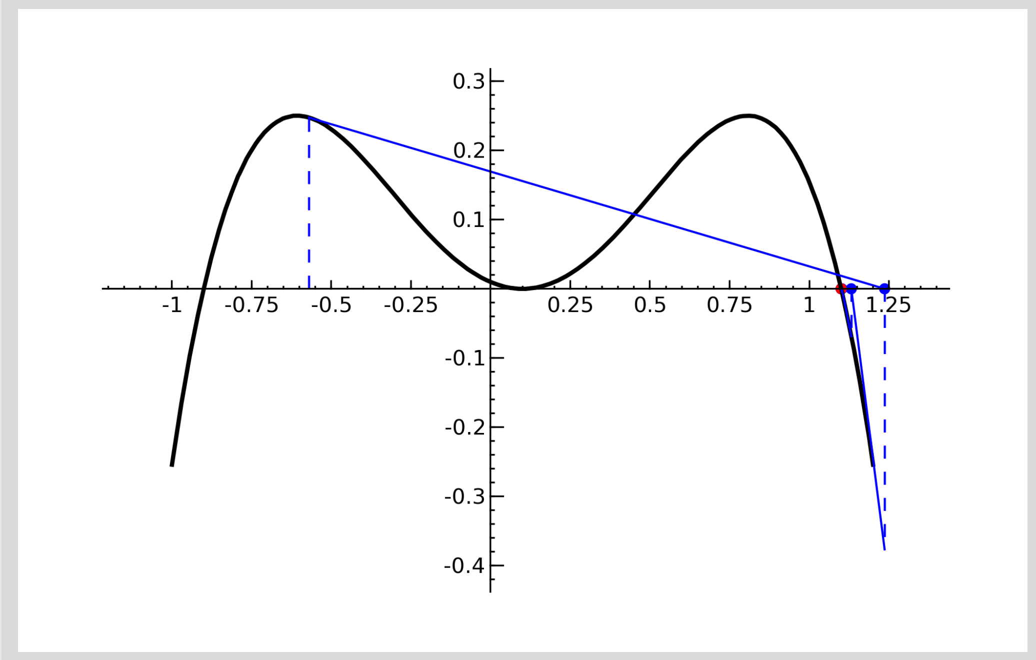
- If we encounter a point whose tangent line is horizontal, the method fails because the line has no root.
- If the method leads us into a "bowl", there may be no way out and towards the zero. The algorithm may "bounce around" in bad estimates.



Newton's Method: Dangers

There are several cautions to observe with Newton's Method.

- If we encounter a point whose tangent line is horizontal, the method fails because the line has no root.
- If the method leads us into a "bowl", there may be no way out and towards the zero. The algorithm may "bounce around" in bad estimates.
- A bad starting point can lead to a root different than the one desired, or even to no root at all.



Newton's Method: Simple example

We will conclude with a simple example that you should work by hand, on paper. In this example, we will estimate a root of $\cos x - x$, correct to the hundredths place, starting at $x = 0$.

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Compute the equation of the first tangent line.

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Compute the equation of the first tangent line. You should get

$$y = -x + 1.$$

(Why? The slope is $f'(0) = -1$. Substitute into $y - y_0 = m(x - x_0)$, where $x_0 = 0$ and $y_0 = f(x_0) = 1$. Solve for y.)

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Now solve for the root of the tangent line.

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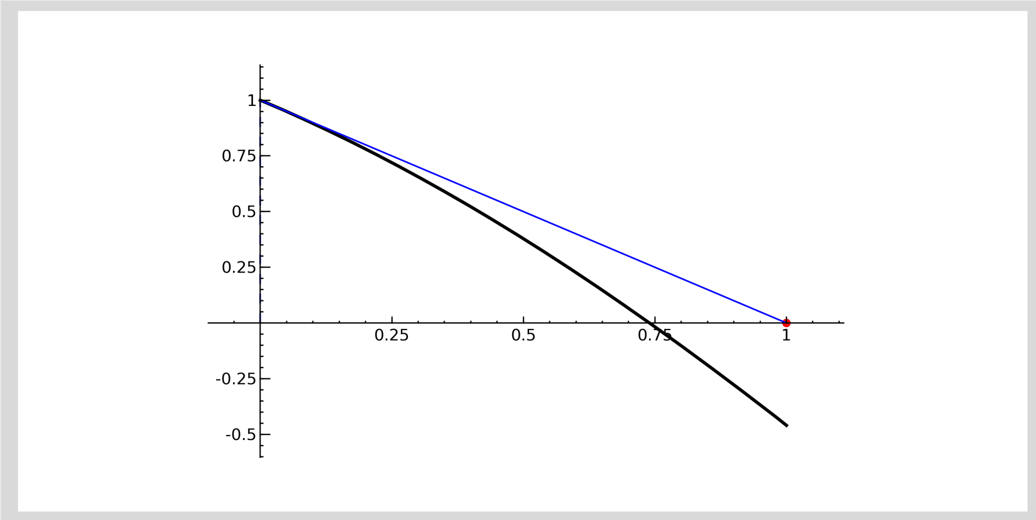
$$y = -x + 1.$$

Now solve for the root of the tangent line. You should get

$$x = 1.$$



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Compute the equation of the second tangent line.

Newton's Method: Simple example

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Compute the equation of the second tangent line. You should get

$$y = -(\sin 1 - 1)(x - 1) + (\cos 1 - 1) \approx -1.841x + 1.382.$$

Now solve for the root of the tangent line.

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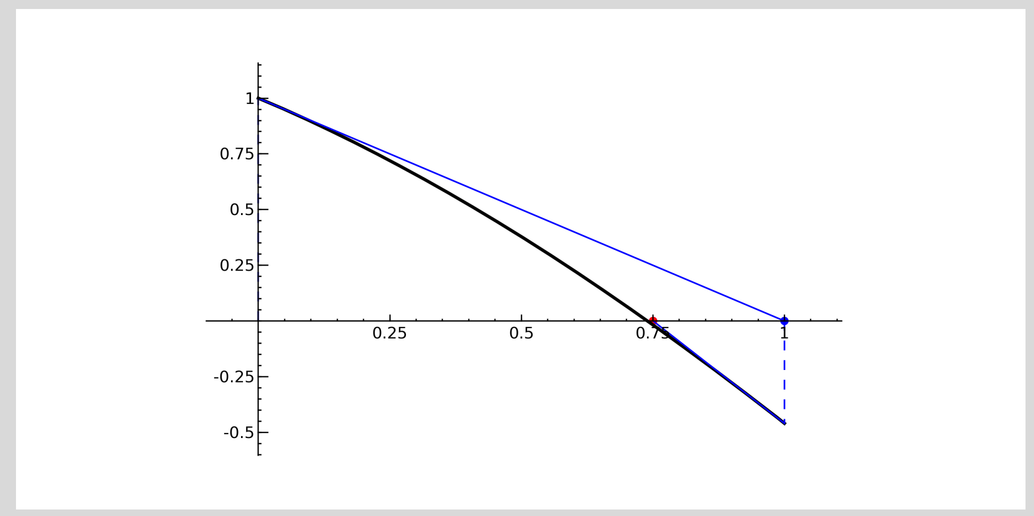
$$y = -(\sin 1 - 1)(x - 1) + (\cos 1 - 1) \approx -1.841x + 1.382.$$

Now solve for the root of the tangent line. You should get

$$x \approx 0.751.$$



Newton's Method: Simple example

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Compute the equation of the third tangent line.

Newton's Method: Simple example

We will conclude with a simple example that you should work by hand, on paper. In this example, we will estimate a root of $\cos x - x$, correct to the hundredths place, starting at $x = 0$.

Compute the equation of the third tangent line. You should get

$$y = -(\sin 0.75 - 1)(x - 0.75) + (\cos 0.75 - 0.75) \approx -1.682x + 1.243.$$

Now solve for the root of the tangent line.

Newton's Method: Simple example

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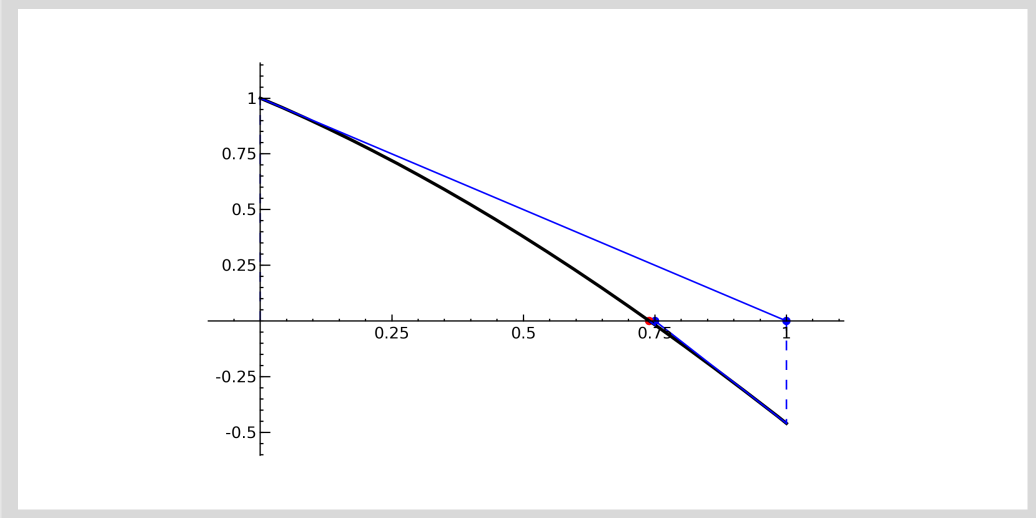
Now solve for the root of the tangent line. You should get

$$x \approx 0.739.$$



Newton's Method: Simple example

We will conclude with a simple example that you should work by hand, on paper. In this example, we will estimate a root of $\cos x - x$, correct to the hundredths place, starting at $x = 0$.

Compute the equation of the fourth tangent line. (Don't fret; we're almost there!)

Newton's Method: Simple example

We will conclude with a simple example that you should work by hand, on paper. In this example, we will estimate a root of $\cos x - x$, correct to the hundredths place, starting at $x = 0$.

Compute the equation of the fourth tangent line. You should get

$$y = -(\sin 0.739 - 1)(x - 0.739) + (\cos 0.739 - 0.739) \approx -1.674x + 1.237.$$

Now solve for the root of the tangent line.

Newton's Method: Simple example

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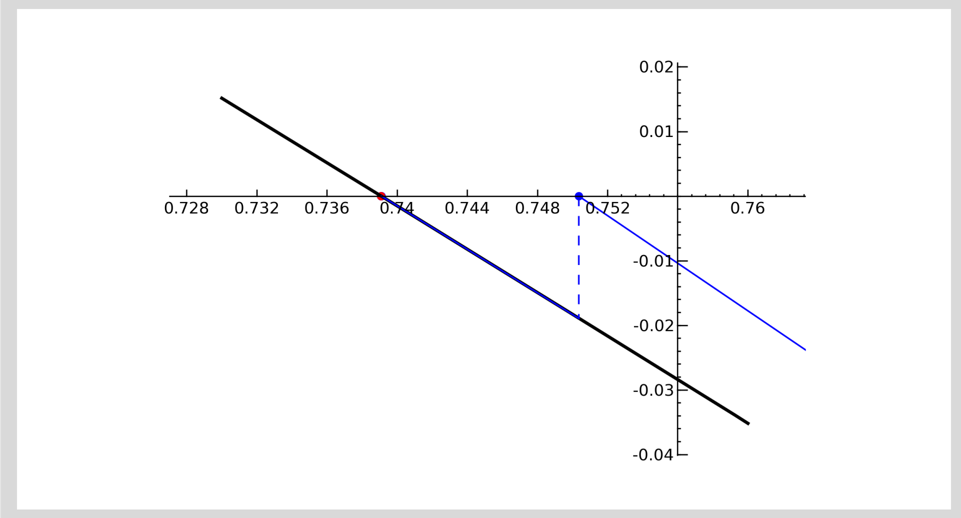
Now solve for the root of the tangent line. You should get

$$x \approx 0.739.$$



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Now solve for the root of the tangent line. You should get

$$x \approx 0.739.$$

The third and fourth step agree in the hundredths place; the root is approximately 0.74.

Conclusion

Newton's method provides a simple method of approximating the roots of functions. It is based on the observation that, in the neighborhood of the root of a function, a line tangent to the function leads one to the root.

- Compute the line tangent to f at $x = a$.
- Find the root of this equation, and call it b.
- Is b "close enough" to the actual root?
  - If not, repeat the process using b in place of a.
  - If so, you are done!

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the **⟨Print⟩** icon, and then saving or printing the pdf file.

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The University of
Southern Mississippi