Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Continuous functions

In mathematics we give a special pride of place to *functions*. Continuous functions possess a special property that make it possible to do calculus. Most of the functions we study in calculus are continuous everywhere; the rest are continuous almost everywhere. In this module we will

- define continuity,
- give examples of continuous as well as discontinuous functions, and
- take a brief glance at an application of continuity.

Defining the problem

In previous modules we have seen that

$$\lim_{x\to 1} x^2 = 1^2 = 1.$$

Notice that you could find this answer simply by substituing $x=1$ into the function $f(x)=x^2$. Similarly, either the algebraic, graphical, or numerical approaches will show that

$$\lim_{x\to 4}(x^3-4x^2+3) = 4^3-4\cdot 4^2+3 = 3$$
$$\lim_{x\to 0}|x+2| = |0+2| = 2$$

and many other limits can be found simply by substituting the x value.

On the other hand, many limits *cannot* be found simply by substituting the x value. You have seen this with

$$\lim_{x\to 0}\sin\left(\frac{1}{x}\right) \qquad \text{or} \qquad \lim_{x\to 1}\frac{x^2-x}{x-1}.$$

What we would like to know is, *which* functions enjoy the property that we can find the limit simply by susbstituting the x value?

Redefining the problem

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We can turn this question into a definition.

Continuity at a point

Definition: *(Continuity at a point)*
A function f is continuous at a point $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

That is, a function is continuous at $x = a$ if we can find the limit by evaluating $f(a)$.

People like to say that a function is continuous at $x = a$ if it has no holes, jumps, or asymptotes at $x = a$.

- If it has a hole at $x = a$, then $f(a)$ is undefined.
- If it has a jump at $x = a$, then $f(a)$ might be defined, but the left- and right-handed limits disagree, so $\lim_{x\to a} f(x)$ does not exist.
- If it has an asymptote at $x = a$, then the limit is not a finite number. (We discuss this in a subsequent module.)

Either way, $\lim_{x\to a} f(x) \neq f(a)$.

Some examples

As mentioned already, $f(x)=x^2$ is continuous at $x=1$ because

$$\lim_{x\to 1} f(x)=f(1).$$

On the other hand, $s(x)=\sin\left(\frac{1}{x}\right)$ is not continuous at $x=0$ because

$$\lim_{x\to 0} s(x)\neq s(0);$$

in fact, $s(0)$ does not even exist!

Definition: If a function f is not continuous at $x=a$ we say that f is discontinuous at $x=a$.

Which functions are continuous on their domains?

You should remember the following fact.

Theorem: The following functions are continuous at every point in their domain:

- polynomials;
- rational functions
  (these have the form $p(x)/q(x)$ where p,q are polynomials);
- exponential functions (a^x where $a>0$ and $a\neq 1$);
- logarithmic functions ($\log_a x$ where $a>0$ and $a\neq 1$);
- trigonometric functions;
- sums, differences, and products of functions named above;
- quotients $f(x)/g(x)$ of functions named above,; and
- composition $f(g(x))$ of functions named above, *provided that the range of g is a subset of the domain of f.*

Example

The following functions are continuous everywhere *on their domains*.

- x^3-4x^2+3, because it is a polynomial;
- $(x^2-x)/(x-1)$, because it is a rational function;
- $\sin(x)/(x^2+1)$, because it is the quotient of a trigonometric function and a polynomial (the domain of the function is $\mathbb{R}$); and
- $\sin(x)/(x^2-1)$, because it is the quotient of a trigonometric function and a polynomial (the domain of the function is $\mathbb{R}\setminus\{-1,1\}$); and
- $\sin\left(x^3-4x^2+3\right)$, because it is the composition of a trigonometric function and a polynomial, *and* the range of x^3-4x+3 is a subset of the domain of $\sin(x)$.

Isn't that everything?

Not all functions are continuous everywhere. An important function that you should know is the greatest integer function, also called the floor function,

$$\lfloor x\rfloor = \text{the largest integer less than or equal to } x.$$

For example,

$$\lfloor 3.9\rfloor = 3 \qquad \text{and} \qquad \lfloor -3.9\rfloor = -4.$$

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Why is the floor function discontinuous? Estimate numerically $\lim_{x\to 0^+} \lfloor x \rfloor$ and $\lim_{x\to 0^-} \lfloor x \rfloor$. You will find that the limits from each side are different. This means that *the limit does not exist, even though the y value exists!* That makes it impossible for $\lim_{x\to 0} \lfloor x \rfloor = \lfloor 0 \rfloor$.

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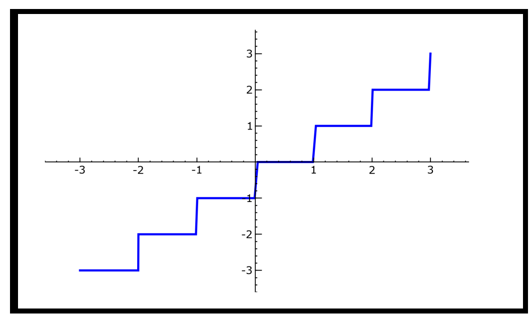
The domain of the floor function is $\mathbb{R}$, so the floor function is *not* continuous everywhere on its domain.

SAGE and discontinuous functions

Discontinuous functions are a weakness for SAGE (and for every computer algebra system). SAGE is not preprogrammed to look for discontinuities when you try to plot its graph.

The following is the output of the command `plot(floor(x),-3,3,thickness=2)`:



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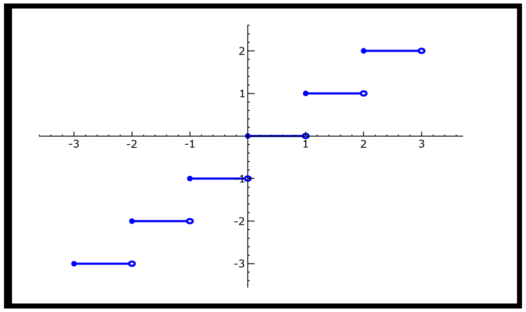
A more accurate graph would look like this:



If a function is discontinuous, it is *your responsibility* to notice this, and to take the necessary precautions for obtaining a correct graph, limit, etc.

Continuous functions

If a function is continuous at *all* points in an interval I, we say that the function is continuous on I.

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Continuous functions

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For example,

- polynomials are continuous on $\mathbb{R}=(-\infty,\infty)$;
- the function $1/x$ is continuous on $\mathbb{R}\setminus\{0\}=(-\infty,0)\cup(0,\infty)$; and
- the function $\tan x$ is continuous on

$$\left\{x\neq\frac{\pi}{2}+\pi k : k\in\mathbb{N}\right\}=\cdots\cup\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\cup\cdots.$$

You should check that these are correct.

Application: Intermediate Value Theorem

An important application of continuity is the Intermediate Value Theorem.

Theorem: *(Intermediate Value Theorem)*

If

- the function f is continuous on the interval $[a,b]$, and
- $f(a)\neq f(b)$,

then

- for any y value C between $f(a)$ and $f(b)$,
- there exists a value c such that
  - $a<c<b$, and
  - $f(c)=C$.

Example of the Intermediate Value Theorem

To illustrate the Intermediate Value Theorem, let $f(x) = x^2$.

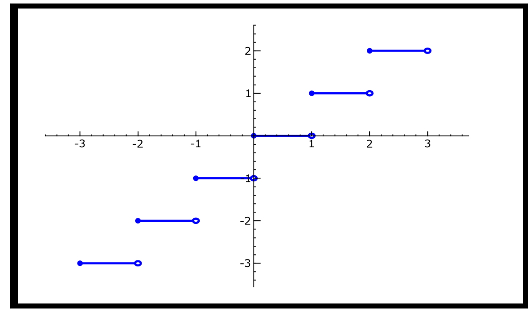
- Suppose $a = 1$ and $b = 2$.
- Since
  - f is continuous,
  - $f(a) = 1 \neq 4 = f(b)$,
  - and $\sqrt{2}$ is between 1 and 4,
- we can conclude that for *some* value $c \in (1,2)$,
- $f(c) = \sqrt{2}$.

We will use this fact in another module to illustrate an application of the Intermediate Value Theorem.

Application: Intermediate Value Theorem

Effectively, the Intermediate Value Theorem says that if f is continuous, then a particle that travels along the curve defined by f does not have to "jump" to move between any two points of the curve.

This is not true for discontinuous functions! Using the floor function, a particle has to "jump" to get from the point $(-0.1,-1)$ to the point $(0,0)$.



Conclusion

- A function is *continuous* if we can compute the two-sided limit simply by substituting the x value into the function.
- Not all functions are continuous everywhere on their domains. The greatest-integer, or floor function, is an example of this.
- An application of continuity is the Intermediate Value Theorem. We will use this in a future module.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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