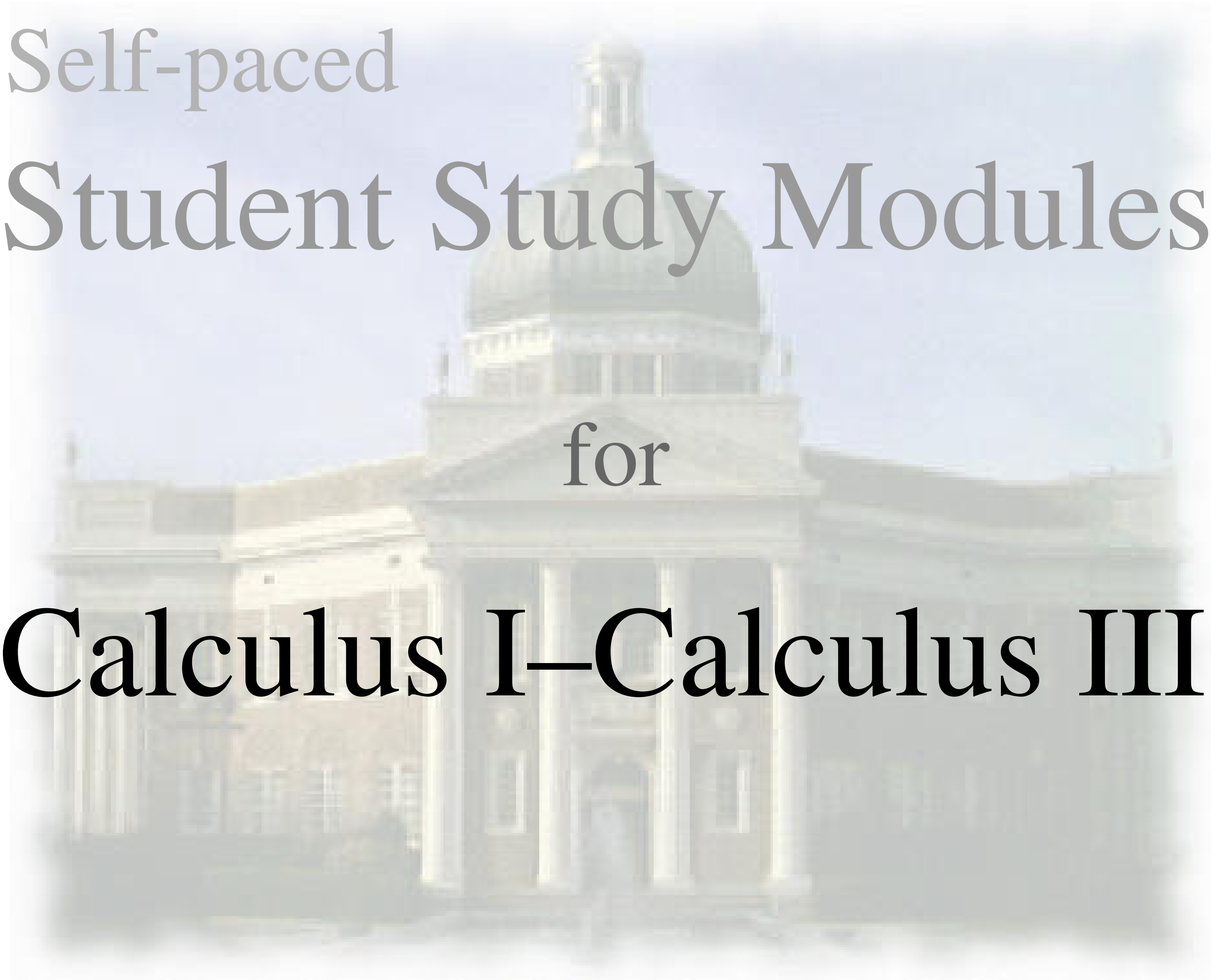

Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Applications of the Intermediate Value Theorem

The Intermediate Value Theorem allows us to conclude that a continuous function takes certain values on an interval. This opens the door to a number of applications. We will explore some of these applications in this module.

Sample problems

Question:

- Approximate a solution (if any) to $\cos(x) = x$ to the thousandths place.
- Approximate the value of $\sqrt[5]{2}$ to the ten-thousandths place.

SAGE worksheets

For this module you will need only a blank SAGE worksheet.

The Intermediate Value Theorem

First let's recall the Intermediate Value Theorem (IVT).

Theorem: *(Intermediate Value Theorem)*

If

- the function f is continuous on the interval $[a, b]$, and
- $f(a) \neq f(b)$,

then

- for any y value C between $f(a)$ and $f(b)$,
- there exists a value c such that
  - $a < c < b$, and
  - $f(c) = C$.

First example

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> **Question:** Approximate a solution to $\cos(x) - x = 0$, if any such solution exists.

Now the question is asking us not to find where two functions are equal, but to find a value of a function $f(x) = \cos(x) - x$. This fits more easily into the IVT point of view.

First example

To apply IVT, we have to make sure that all the hypotheses are met. The first hypothesis is that the function $f(x) = \cos(x) - x$ is continuous. You have learned in class that

- $y = \cos(x)$ is a continuous function everywhere;
- $y = x$ is a continuous function everywhere; and
- sums and differences of continuous functions are continuous on the intervals where both functions are continuous.

Thus $\cos(x) - x$ is continuous everywhere.

You must check for continuity anytime you wish use IVT.

First example

Now that we've verified the continuity of the function, we can try to see if there is any value of x such that $\cos(x) - x = 0$.

How does IVT help? If we can find

- one value of x such that $\cos(x) - x$ is negative, and
- one value of x such that $\cos(x) - x$ is positive, then

IVT assures us that some x value *between* those two values has a y value of zero. How?

First example

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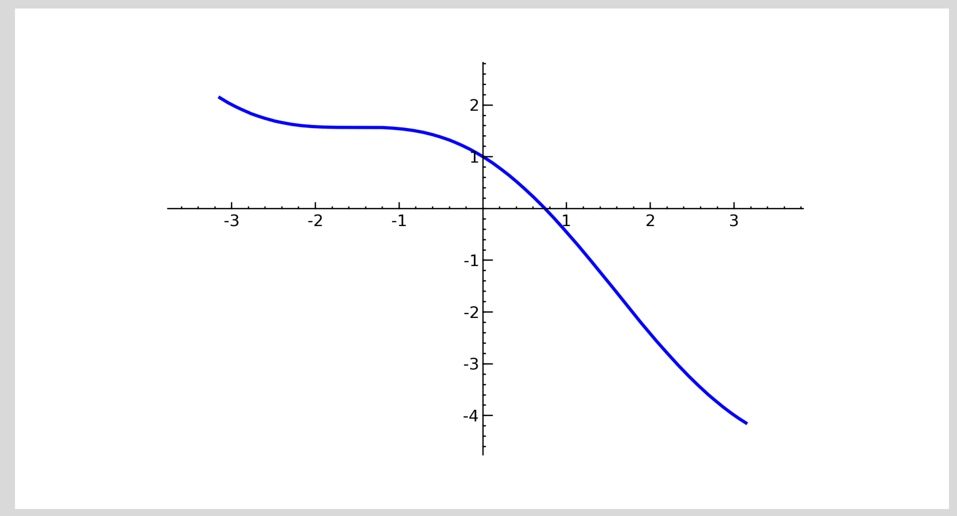
We obtained the following graph by plotting on the interval $[-\pi, \pi]$:



This reveals the following:

- $f(0)$ is positive;
- $f(1)$ is negative.

(You can approximate their values as 1 and -0.4597.)

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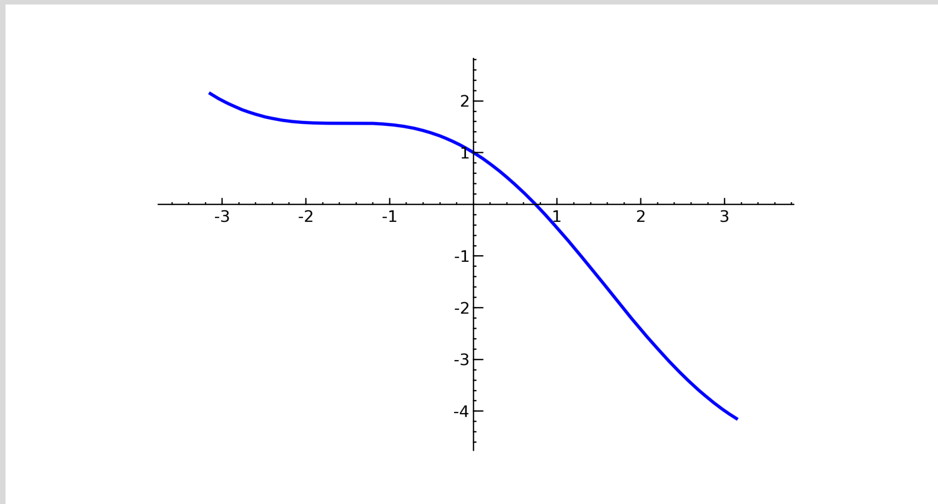


This reveals the following:

- $f(0)$ is positive;
- $f(1)$ is negative.

(You can approximate their values as 1 and -0.4597.)

By the IVT, we know that for *some* value c between $a = 0$ and $b = 1$, $f(c) = 0$.

First example

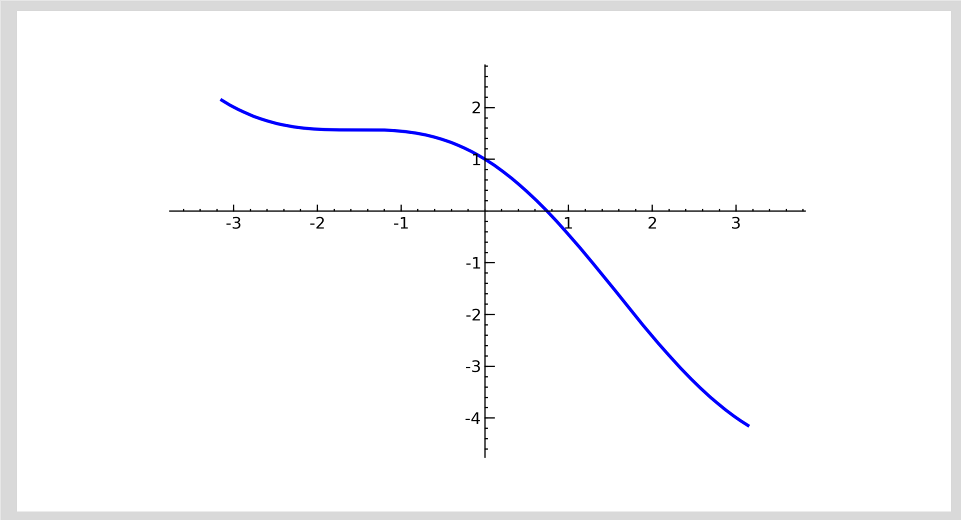
Why did we need IVT to draw this conclusion? *Continuity.* If we did not know that the function is continuous, we would have to worry about holes. Graphing calculators and computers rarely, if ever, display holes in the graphs of their functions.

First example

Now we know that there is *some* value c for which $\cos c - c = 0$, and thus $\cos c = c$. However, the problem asked for something more: a value of c, *accurate to the thousandths place.* How can we use IVT to get this kind of accuracy? Redraw the graph with smaller values of a and b!

First example

Look at the previous graph again.



It looks as if the root lies between 0.6 and 0.8. Let's plot f on that smaller interval and see if we get anything different.

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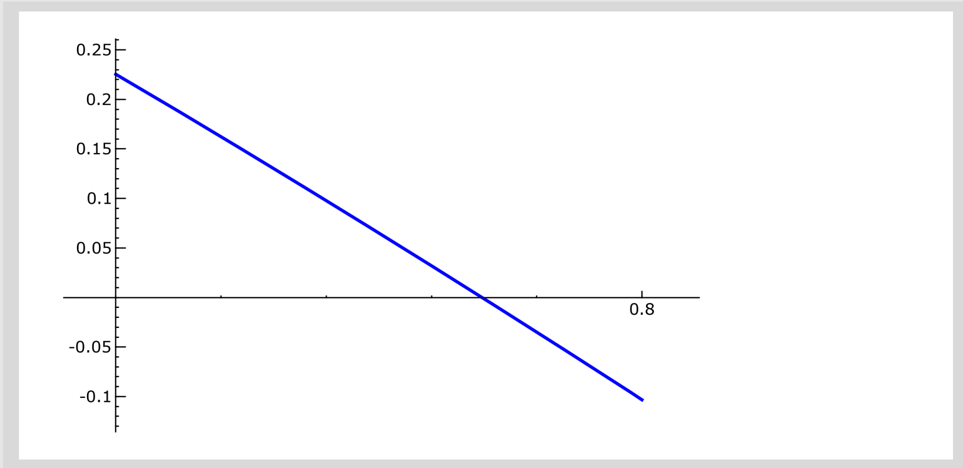

We're getting closer! Now try it on the interval $[0.71, 0.75]$.

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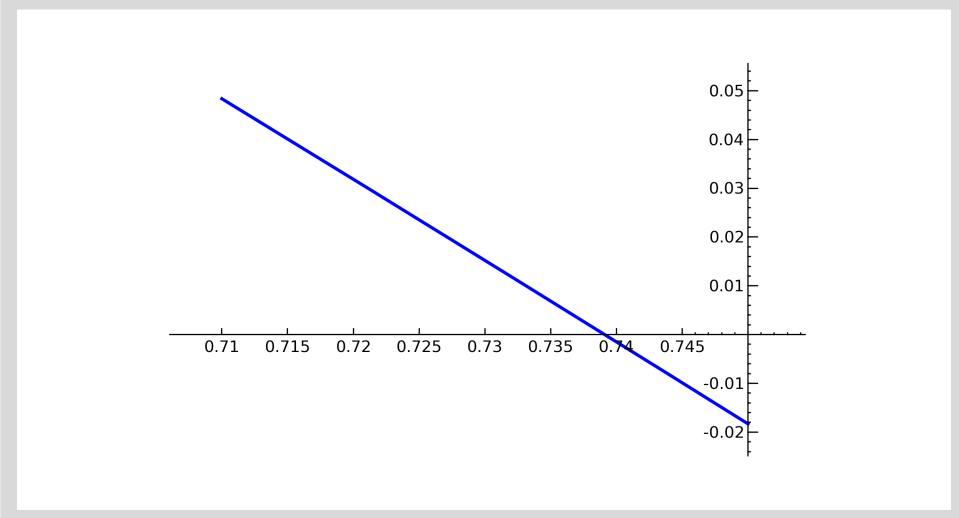

We can see now that the curve intersects the axis between 0.735 and 0.74.

First example

Repeat this process of narrowing the graph until you have an estimate that is accurate to the thousandths place.

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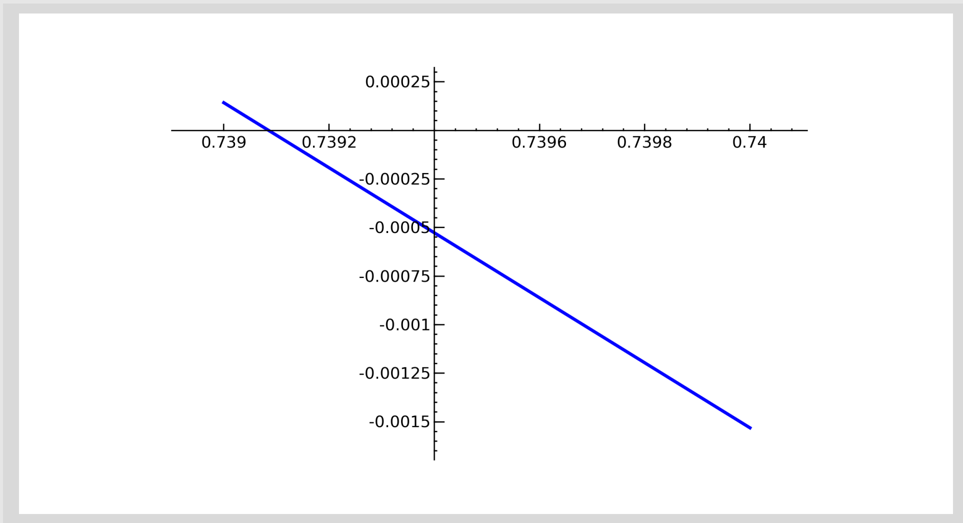
Repeat this process of narrowing the graph until you have an estimate that is accurate to the thousandths place.

How will you know when you're done? When the curve intersects the graph between two values of x that have the same digits up to the thousandths place. The author, for example, obtained this graph:



...using the commands

```
f = cos(x)-x
fplot = plot(f,0.739,0.740)
fplot.show()
```

and concluded that the root is approximately 0.739.

The method of bisection

We can formalize the method shown in the previous example as follows.

- Let (a,b) be an interval and f a function such that $f(a)$ and $f(b)$ have opposite signs (one is positive, the other negative).
- If rounding a and b to the desired precision gives the same number, you are done: the answer is a.
- Let $c = \frac{a+b}{2}$ be the midpoint of $[a,b]$.
- If $f(c)$ has the same sign as $f(b)$, repeat these steps with the interval (a,c).
- If $f(c)$ has the same sign as $f(a)$, repeat these steps with the interval (c,b).

This method will find the approximate value of any root, although it can be very slow.

Second example

A similar procedure will help us approximate the fifth root of two. This problem is a little different, inasmuch as we don't have a function yet! To find the function, let's reason from what we want.

Second example

Question: Approximate the value of $\sqrt[5]{2}$ to the ten-thousandths place.

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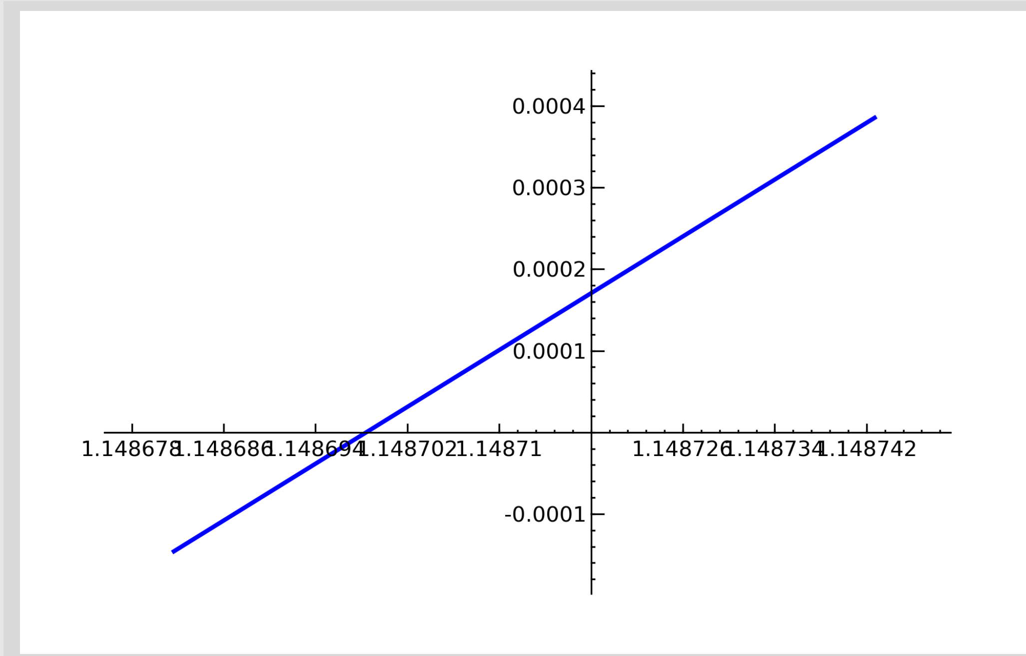
This suggests the function $f(x) = x^5 - 2$. Again, we have to check that this function is continuous, and it is.

Use the method of bisection to approximate the solution to the equation $x^5 - 2 = 0$, accurate to the ten-thousandths digit. You know that $\sqrt[5]{2}$ is larger than 1 and smaller than 2, so start with $a = 1$ and $b = 2$.

Second example

After fourteen steps, we estimated $\sqrt[5]{2} \approx 1.1487$. We had the following graph.



The importance of continuity

We cannot overemphasize the importance of continuity to the IVT.

Suppose, for example, that you wanted to find a root of the function

$$f(x) = \frac{1}{x+1} - \frac{\lfloor x \rfloor}{100} - 1.$$

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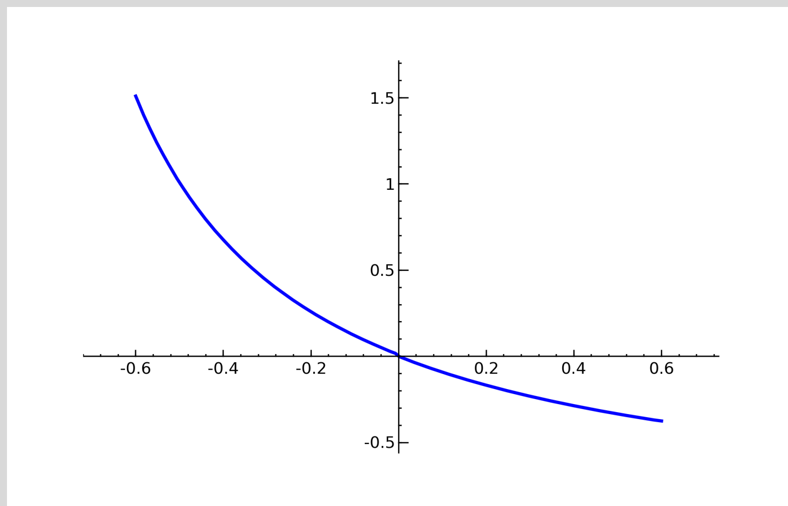

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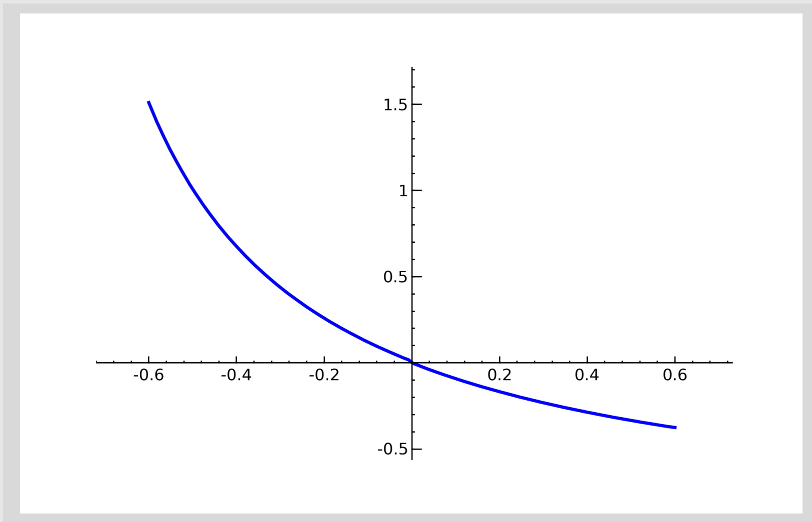

Question: Is there a root at the origin?

No, there is not. It may look as if a root lies there, but the function is discontinuous at $x = 0$ (because $\lfloor x \rfloor$ is discontinuous at $x = 0$).

You might notice this if you zoomed in a little more.

Conclusion

- We can use the Intermediate Value Theorem to approximate solutions to equations and irrational numbers.
- We must remember to check whether a function is continuous before trying to apply the theorem.
- The IVT is not always the best method to solve equations or approximate irrational numbers. Later, you will learn *Newton's Method*, which is usually a much more efficient method.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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