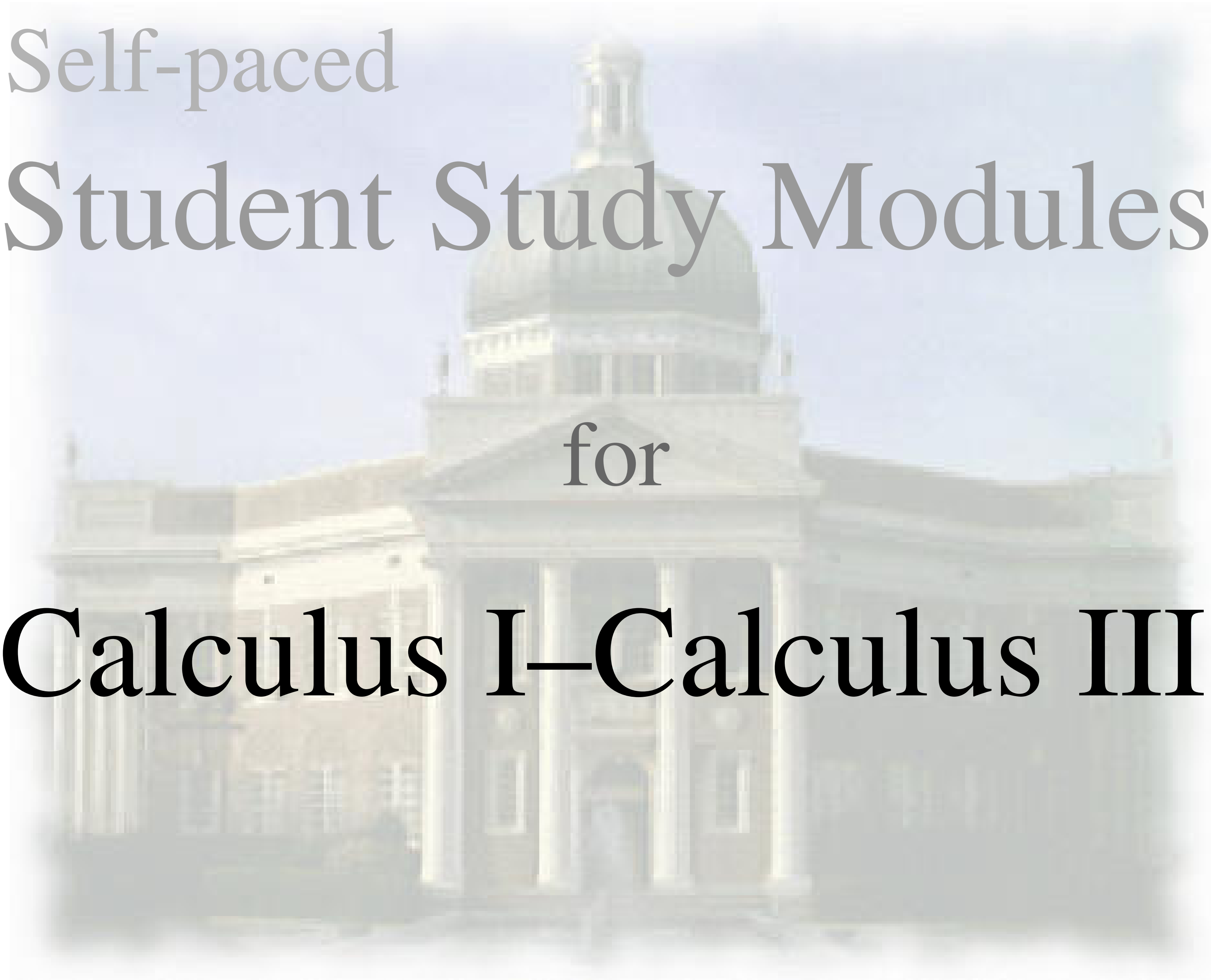

Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Continuity of Functions

Determining whether a function is continuous on a given domain, requires showing that at each point in the domain that the functions is continuous.

- We know that all polynomials are continuous on $\mathbb{R}$, and thus they are continuous on any interval $[a,b] \subset \mathbb{R}$.
- Similarly, the elementary functions such as $\sin(x)$, $\cos(x)$ and $\exp(x)$ are all continuous on $\mathbb{R}$.
- To determine whether a function f is continuous at a point x_0 in the domain, we have to show that the limit of the function as it approaches x_0 from the left equals the limit of the functions as it approaches x_0 from the right, and these must equal the value of the function at x_0. Formally this is given by,

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

  Note that this means the function must be defined at x_0. For example the function $\ln(x)$ is not continuous at $x = 0$ because the logarithm is not defined there.

- If you need to review this, go to ⟨**CalcText**⟩ and search for continuity.

Defining the problem

The Problem: Determine on which domain

$$g(x) = \begin{cases} (x^2-4)/(x+2) & \text{if } x < 0 \\ 5 & \text{if } x = 0 \\ (x^3+2x^2-2x-4)/(x^2+4x+4) & \text{if } x > 0 \end{cases}$$

is continuous. Can the function $g(x)$ be redefined to have a greater domain of continuity?

Defining the problem

The Problem: Determine on which domain

$$g(x) = \begin{cases} (x^2-4)/(x+2) & \text{if } x < 0 \\ 5 & \text{if } x = 0 \\ (x^3+2x^2-2x-4)/(x^2+4x+4) & \text{if } x > 0 \end{cases}$$

is continuous. Can the function $g(x)$ be redefined to have a greater domain of continuity?

What does it mean to have a greater, i.e., a larger domain of continuity?

Defining the problem

The Problem: Determine on which domain

$$g(x) = \begin{cases} (x^2-4)/(x+2) & \text{if } x<0 \\ 5 & \text{if } x=0 \\ (x^3+2x^2-2x-4)/(x^2+4x+4) & \text{if } x>0 \end{cases}$$

is continuous. Can the function $g(x)$ be redefined to have a greater domain of continuity?

What does it mean to have a greater, i.e., a larger domain of continuity?

- If $g(x)$ is continuous on the set $S \subset \mathbb{R}$, to have a greater domain of discontinuity means that the set of points on which $g(x)$ is continuous can be extended to a larger set $\tilde{S}$ with $S \subset \tilde{S}$, such that there is a function, we'll call it $\tilde{g}(x)$, that is continuous on on $\tilde{S}$, *and* $\tilde{g}(x) = g(x)$ for all $x \in S$.
- First, if $g(x)$ is continuous on $\mathbb{R}$, then $S = \mathbb{R}$, and it is impossible to increase the size of this set.

Are there any other points of discontinuity?

Examining the function

- The possible points of discontinuity are when $x = 0$, (the break point in the definition of $g(x)$), and at the points at which the denominators are zero, i.e., where $(x+2) = 0$, and when $(x^2+4x+4) = 0$.
- The first of equations can be solved immediately to give $x = -2$, while the second requires only a bit more work in factoring the quadratic polynomial into the terms $(x+2)(x+2)$. Thus $x = -2$ and $x = 2$ are possible points where $g(x)$ may fail to be defined. Thus,

$$g(x) = \begin{cases} (x^2-4)/(x+2), \text{ if } x < 0 & \Longrightarrow \text{at } x = -2 \\ 5, \text{ if } x = 0 & \Longrightarrow \text{at } x = 0 \\ (x^3+2x^2-2x-4)/(x^2+4x+4), \text{ if } x > 0 & \Longrightarrow \text{at } x = -2 \end{cases} \quad \begin{matrix} \text{Possible point of discontinuity} \end{matrix}$$

Examining the function

- The possible points of discontinuity are when $x = 0$, where there is a break point in the definition of $g(x)$, and at the points at which the denominators are zero, i.e., where $(x+2) = 0$, and when $(x^2+4x+4) = 0$.
- The first of equations can be solved immediately to give $x = -2$, while the second requires only a bit more work in factoring the quadratic polynomial into the terms $(x+2)(x+2)$. Thus $x = -2$ and $x = 2$ are possible points where $g(x)$ may fail to be defined. Thus,

$$
\begin{array}{ll}
 & \text{Possible point of discontinuity} \\
g(x) = \begin{cases} (x^2-4)/(x+2), \text{ if } x < 0 \\ 5, \text{ if } x = 0 \\ (x^3+2x^2-2x-4)/(x^2+4x+4), \text{ if } x > 0 \end{cases} & \begin{array}{l} \Longrightarrow \text{at } x = -2 \\ \Longrightarrow \text{at } x = 0 \\ \Longrightarrow \text{at } x = -2 \end{array}
\end{array}
$$

We should graph each branch of the function $g(x)$ so that we have a sense of what his going on.

Plotting of $g(x)$

- We graph the the branches of the function, $f(x) = g(x)$, i.e.,
  $g_1(x) = (x^2 - 4)/(x+2)$,
  $g_2(x) = 5$, and
  $g_3(x) = (x^3 + 2x^2 - 2x - 4)/(x^2 + 4x + 4)$
  using ⟨**Plot**⟩ to input the value `(x**3 + 2*x**2 - 2*x -4)/(x**2 + 4*x + 4), 5, (x*x-4)/(x+2)` where the independent variable is `x`.

You will need to examine the figure you obtain closely. Try to understand what each line represents, and how they relate to the three branches of the function $g(x)$. Try to identify the possible singularities, and points of discontinuity of the function $g(x)$.

The graph of $g(x)$

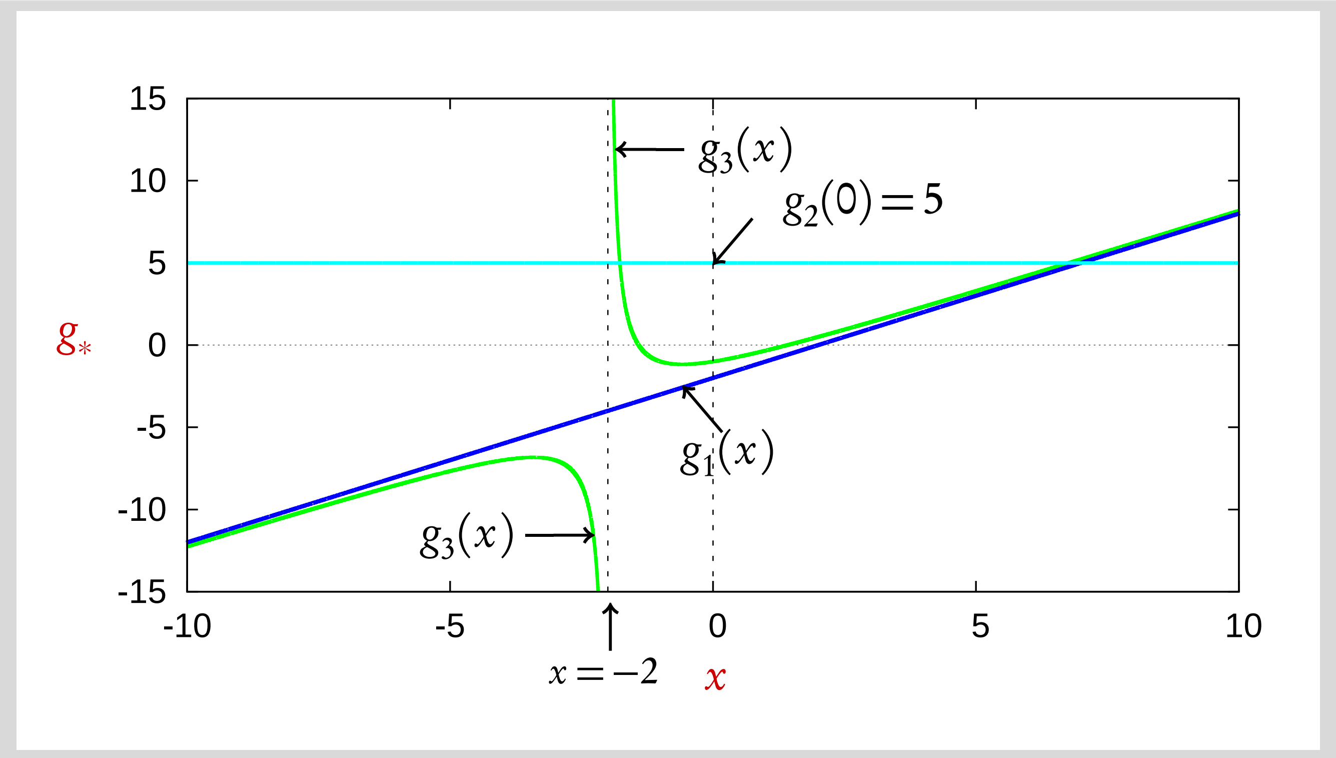

Figure 1: Note that function $g(x) = g_1(x)$ for all $x < 0$ appears to be the equation of a line, i.e., a function of the form $y = mx + b$, and that the function $g_3(x)$ has a singularity at $x = -2$. The value of $g(x)$ at $x = 0$, i.e., $g(0)$, is a single point that is not connected to the values of $g(x)$ to the left or to the right of this point.

Examining $g(x)$ on $(-\infty, 0)$

- If $x < 0$ then:

$$g(x) = \frac{(x^2-4)}{(x+2)} = \frac{(x-2)(x+2)}{(x+2)} = g_1(x).$$

- This function is not continuous since the function is not defined at $x = -2$. However, since

$$\lim_{x\to -2^-} g(x) = \frac{(x-2)(x+2)}{(x+2)} = \lim_{x\to -2^-} (x-2) = -4 \quad (1)$$

$$\lim_{x\to -2^+} g(x) = \frac{(x-2)(x+2)}{(x+2)} = \lim_{x\to -2^+} (x-2) = -4 \quad (2)$$

the limit from the left equals the limit from the right. This is typical of a removable discontinuity.

- Thus, if we define $\tilde{g}(x) = (x-2)$, then $\tilde{g}(x) = g(x)$ on $(-\infty, 0)$ except at $x = -2$, and $\tilde{g}$ is continuous on $(-\infty, 0)$.

Note that $g_1(x) = x - 2$ is the equation of a line, as we observed in the graph on page 3

The details of taking the limit

- The evaluation the limit from below $\lim_{x\to -2^-} \frac{(x-2)(x+2)}{(x+2)} = -4$, in (1) and similarly the limit in (2) seems obvious. We simply cancel the terms $\frac{(x+2)}{(x+2)} = 1$.

- There is a technical issue here, i.e., at the limit point, $x = -2$, we have that $\frac{(x+2)}{(x+2)} = \frac{0}{0}$, and we know that we cannot evaluate this indeterminate form.
- We can argue that for each x_0 in any neighborhood of $x = -2$ we have that $(x_0 + 2)/(x_0 + 2) = 1$ and thus the limit must be 1, and thus in any neighborhood of $x = -2$ the value of $\frac{(x-2)(x+2)}{(x+2)} = 1 \cdot (x-2) - 4$.

  As long as we are not at $x = -2$, we can evaluate the expression $(x+2)/(x+2) = 1$, and this is all that matters in taking the limit.

Examining $g(x)$ on $(0,\infty)$

- If $x > 0$ then
$$g(x) = (x^3+2x^2-2x-4)/(x^2+4x+4) = \frac{(x^2-2)(x+2)}{(x+2)^2} = \frac{x^2-2}{x+2}$$
and $g(x)$ has no singularities on the domain $(0,\infty)$ since the point $x=-2$, i.e., where the denominator vanishes, is outside the domain of definition of this branch of the function.

- Note that
$$(x^3+2x^2-2x-4)/(x^2+4x+4) = \frac{x^2-2}{x+2}$$
represent the same functions on $(0,\infty)$, i.e., at each point $x \in (0,\infty)$ both of these are algebraically equivalent yielding the same value.

- Thus $g(x)$ is continuous on $(0,\infty)$ since $\frac{(x^2-2)(x+2)}{(x+2)^2}$ is a rational function with no singularities in it its domain.

Examining $g(x)$ in a neighborhood of $x = 0$

- If $x = 0$, then consider
$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} x - 2 = -2,$$
and
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{x^2 - 2}{x + 2} = -1.$$

- Thus $\lim_{x \to 0^-} g(x) \neq \lim_{x \to 0^+} g(x)$ and $g(x)$ is not continuous at $x = 0$. This is not a removable discontinuity since we cannot redefine $g(0)$ in a manner which makes the limit from the left equal the limit from the right, and which also makes this limit equal to some value of g that we can assign at $x = 0$.

Examining $g(x)$ in a neighborhood of $x=0$

- If $x=0$, then consider
$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x-2 = -2,$$
and
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{x^2-2}{x+2} = -1.$$

- Thus $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$ and $g(x)$ is not continuous at $x=0$. This is not a removable discontinuity since we cannot redefine $g(0)$ in a manner which makes the limit from the left equal the limit from the right, and which also makes this limit equal to some value of g that we can assign at $x=0$.
- The point $x=0$ is an essential singularity of $g(x)$.

From everything we have done so far, the conclusion is that $g(x)$ as defined can be made continuous on the domain $(-\infty,0)\bigcup(0,\infty)$.

Let's pull this all together.

Redefining $g(x)$

- We have ascertained that $g(x)$ is continuous on $\mathbb{R}$, except at $x=-2$, and $x=0$.
- We showed that if $g(x)$ is redefined to be $\tilde{g}(x)$ on $(-\infty,0)$, then $g(x)$ is continuous on $(-\infty,0)$, but that at $x=0$, there was no way to redefine g so as to make it continuous at that point.
- If we define

$$\tilde{g}(x)=\begin{cases} x-2 & \text{if } x<0 \\ -2 & \text{if } x=0 \\ (x^2-2)/(x+2) & \text{if } x>0 \end{cases}$$

then $\tilde{g}(x)=g(x)$ except at $x=-2$ and $x=0$ and $\tilde{g}$ is continuous on $(-\infty,0]\bigcup(0,\infty)$.
- If we define

$$\overline{g}(x)=\begin{cases} x-2 & \text{if } x<0 \\ -1 & \text{if } x=0 \\ (x^2-2)/(x+2) & \text{if } x>0 \end{cases}$$

then $\overline{g}(x)=g(x)$ except at $x=-2$ and $x=0$ and $\overline{g}$ is continuous on $(-\infty,0)\bigcup[0,\infty)$.
- There is no way to redefine g to be continuous on $(-\infty,\infty)$, and thus $\tilde{g}$ or $\overline{g}$ are the only continuous extensions of the function g.

Conclusion

- Some functions can be *extended* in such a way that some, or even all, of their discontinuities are "removed".
- To determine which discontinuities are *removable* and which are *essential*, we must examine the limit from each side: from the left or from the right.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the **⟨Print⟩** icon, and then saving or printing the pdf file.

Department of Mathematics at
The University of
Southern Mississippi