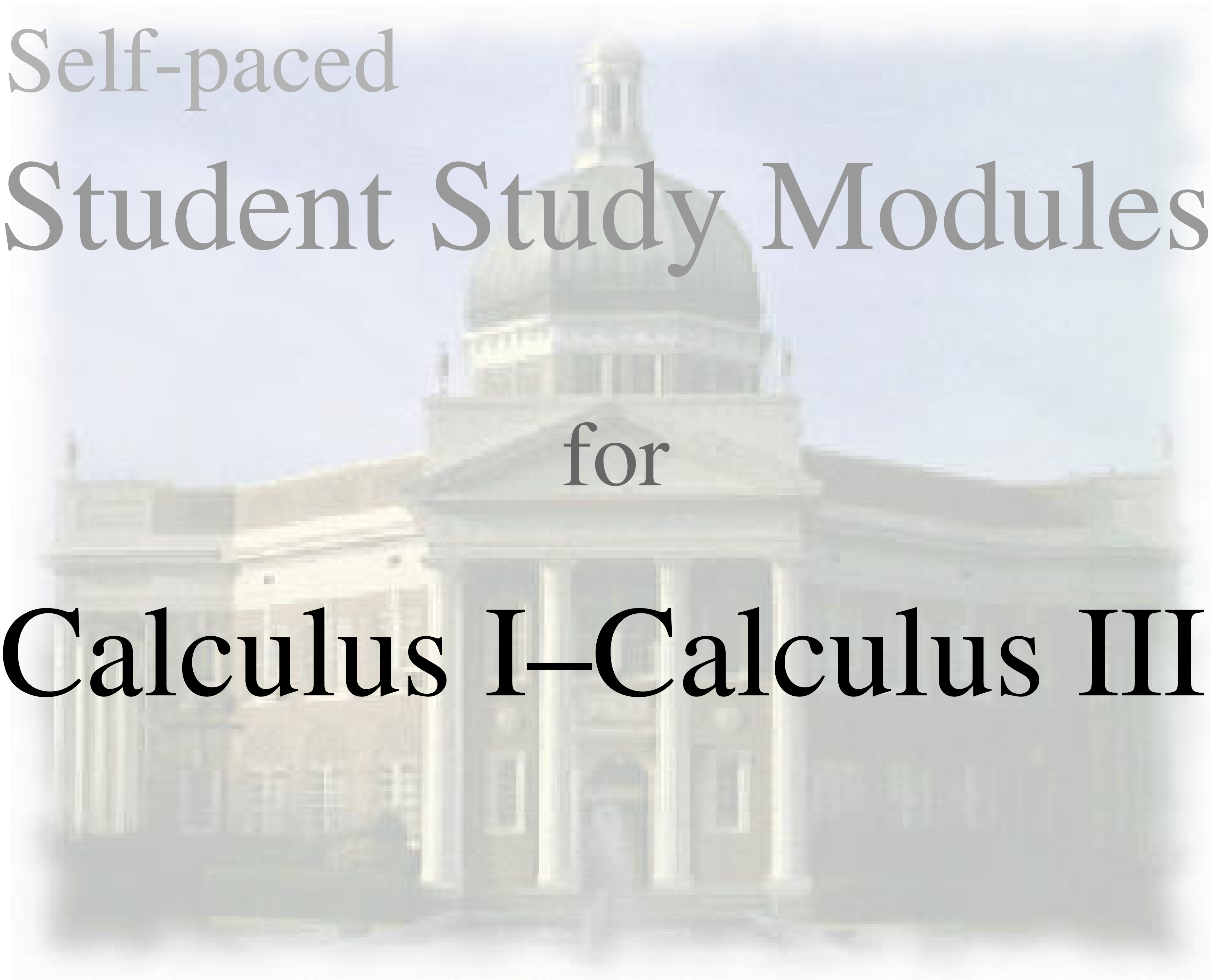

Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages **⟨Next⟩**, backward **⟨Prev⟩**, or view all the slides in this tutorial **⟨Index⟩**.
- The **⟨Back to Calc I⟩** button returns you to the course home page.
- A full symbolic algebra package **⟨Sage⟩** is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text **⟨CalcText⟩** provides a quick search of basic calculus topics.
- You can get help from Google Calculus **⟨GoogleCalc⟩**.
- A monochrome copy of this module is suitable for printing **⟨Print⟩**.

When all else fails, feel free to contact your instructor.

Related Rates

The derivative expresses the *instantaneous rate of change*. We can use this fact to solve problems involving rates of change that are related to each other.

Defining the problem

Question:

- The radius of a ripple in a pond grows at a rate of 5 cm/s. How fast is the circumference changing at 3 seconds?
- The radius of a ripple in a pond grows at a rate of 5 cm/s. How fast is the area changing at 3 seconds?
- A camera is directed at the launch of a rocket. The camera is 3 km from the rocket. Once the rocket rises 1 km, it has a speed of 1500 km/hr. How fast must the angle of the camera change to remain fixed on the rocket?

Leibniz notation

In most modules we have favored Newton's notation f' for the derivative of a function f. In this module we will use Leibniz notation df/dt instead. *Why?* Clarity of expression. All the problems can be solved using Newton's notation, but Newton's notation does not indicate the independent variable. Problems in related rates usually have several variables that depend on time, and the derivatives are always *with respect to time*, which the Leibniz notation df/dt emphasizes.

First example: Experimentation

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the circumference changing at 3 seconds?

The problem asks us to find dC/dt, given that $dr/dt = 5$.

First example: Experimentation

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the circumference changing at 3 seconds?

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You can visualize this using the SAGElet “The Ripple Effect: Circumference”. It animates the ripple, and approximates dC/dt using the secant lines. Experiment with different values of dr/dt (“change in radius per second”) and see if you can determine a pattern.

First Example: Experimentation

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the circumference changing at 3 seconds?

The problem asks us to find dC/dt, given that $dr/dt = 5$.

You can visualize this using the SAGElet "The Ripple Effect: Circumference". It animates the ripple, and approximates dC/dt using the secant lines. Experiment with different values of dr/dt ("change in radius per second") and see if you can determine a pattern.

Interestingly, any fixed choice of dr/dt gives a constant value for dC/dt, *regardless of the value of* r. Here are some values that we found:

$\frac{dr}{dt}$	0	1	5	10
$\frac{dC}{dt}$	0	2π	10π	20π

This suggests the relationship $dC/dt = 2\pi(dr/dt)$. Since dr/dt is constant, so is dC/dt. But can we *prove* it?

First Example: Concept

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the circumference changing at 3 seconds?

The problem asks us to find dC/dt, given that $dr/dt = 5$.

If we had a relationship between r and C, we could differentiate it with respect to t. Since the radius r and the circumference C depend on time t, the Chain Rule would give us quantities dr/dt and dC/dt. We would end up with a relationship between the rates!

First Example: Concept

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the circumference changing at 3 seconds?

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If we had a relationship between r and C, we could differentiate it with respect to t. Since the radius r and the circumference C depend on time t, the Chain Rule would give us quantities dr/dt and dC/dt. We would end up with a relationship between the rates!

This is the general procedure for related rates:

- find an equation that relates the quantities;
- differentiate with respect to time; and
- solve for the unknown rate.

First Example: Solution

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the circumference changing at 3 seconds?

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An equation that relates r and C is

$$C = 2\pi r.$$

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Differentiating with respect to time gives

$$C = 2\pi r$$

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}.$$

First Example: Solution

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This is the general procedure for related rates:

- find an equation that relates the quantities;
- differentiate with respect to time; and
- solve for the unknown rate.

Substituting $dr/dt = 5$, we have

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$
$$\frac{dC}{dt} = 10\pi.$$

Since t does not appear in the equation, $dC/dt = 10\pi$ for *all* values of t. At 3 seconds, therefore, the circumference is increasing at 10π cm/s, approximately. This agrees with the answer found by experimentation.

Second Example: Experimentation

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

The problem asks us to find dA/dt, given that $dr/dt = 5$.

Second Example: Experimentation

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

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You can visualize this using the SAGElet "The Ripple Effect: Area". It animates the ripple, and approximates dA/dt using the secant lines of A. Experiment with different values of dr/dt ("change in radius per second") and see if you can determine a pattern.

Second Example: Experimentation

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

The problem asks us to find dA/dt, given that $dr/dt = 5$.

You can visualize this using the SAGElet "The Ripple Effect: Area". It animates the ripple, and approximates dA/dt using the secant lines of A. Experiment with different values of dr/dt ("change in radius per second") and see if you can determine a pattern.

Unlike the circumference problems, fixed values of dr/dt do *not* result in a constant dA/dt, depending on the value of r. Here are some values that we found with $r' = 5$:

t	1	2	3	4	5
r	5	10	15	20	25
dA/dt	25π	75π	125π	175π	225π

Notice that $dA/dt \approx 125\pi$ when $t = 3$. It seems to obey the pattern

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} - \left(\frac{dr}{dt}\right)^2 \pi.$$

(This pattern is not obvious! Don't worry if you couldn't guess it.) But can we *prove* it?

Second Example: Concept

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

The problem asks us to find dA/dt, given that $dr/dt = 5$.

As in the previous example, if we had a relationship between r and A, then we could differentiate it with respect to t and obtain a relationship between the rates dr/dt and dC/dt.

Second Example: Concept

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

The problem asks us to find dA/dt, given that $dr/dt = 5$.

As in the previous example, if we had a relationship between r and A, then we could differentiate it with respect to t and obtain a relationship between the rates dr/dt and dA/dt.

Recall the general procedure for related rates:

- find an equation that relates the quantities;
- differentiate with respect to time; and
- solve for the unknown rate.

Second Example: Solution

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

The problem asks us to find dA/dt, given that $dr/dt = 5$.

Recall the general procedure for related rates:

- find an equation that relates the quantities;
- differentiate with respect to time; and
- solve for the unknown rate.

An equation that relates r and A is

$$A = \pi r^2.$$

Second Example: Solution

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

The problem asks us to find dA/dt, given that $dr/dt = 5$.

Recall the general procedure for related rates:

- find an equation that relates the quantities;
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Differentiating with respect to time gives

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}.$$

Second Example: Solution

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

The problem asks us to find dA/dt, given that $dr/dt = 5$.

Recall the general procedure for related rates:

- find an equation that relates the quantities;
- differentiate with respect to time; and
- solve for the unknown rate.

Substituting $dr/dt = 5$, we have

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 10\pi r.$$

But what is r?

Second Example: Solution

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

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Substituting $dr/dt = 5$, we have

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$$\frac{dA}{dt} = 10\pi r.$$

But what is r?

It can happen when solving a related-rate problem that you need to determine the value of a quantity that was not given. In this case, we know that the radius increases at a rate of 5 cm/s. This is a linear relationship, so $r = 5t$. At this particular moment (3 seconds) $t = 3$, so $r = 15$.

Second Example: Solution

Question: The radius of a ripple in a pond increasing at a rate of 5 cm/s. How fast is the area changing at 3 seconds?

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$$\text{Hence } \frac{dA}{dt} = 150\pi.$$

Second Example: Comparison to approximation

Unlike the example with circumference, the approximation here was somewhat off. Using secant lines, the SAGElet approximated $dA/dt \approx 125\pi$. The correct answer was instead $dA/dt = 150\pi$. The error is due to a term in the pattern we identified with the approximation, $dA/dt \approx 2\pi r(dr/dt) - (dr/dt)^2\pi$. Increasing the value of "units per second" in the SAGElet obtains better approximations, because the secant lines are computed closer to $t = 3$. *(Try it!)* For example, we obtained the following results with different values of "units per second":

t	2	2.25	2.5	2.75	3	3.25	3.5	3.75	4
$n = 1$	75π				125π				175π
$n = 2$	87.5π		112.5π		137.5π		162.5π		187.5π
$n = 4$	93.75π	106.25π	118.75π	131.25π	143.75π	156.25π	168.75π	181.25π	193.75π

Figure 1: Values of dA/dt for different values n of "units per second".

As n increases, the accuracy of the approximation of dA/dt increases.

Third Example

Question: A camera is directed at the launch of a rocket. The camera is 3 km from the rocket. Once the rocket rises 1 km, it has a speed of 1500 km/hr. How fast must the angle of the camera change to remain fixed on the rocket?

The problem asks us to find $d\alpha/dt$, given that $dh/dt = 1500$.

Third Example

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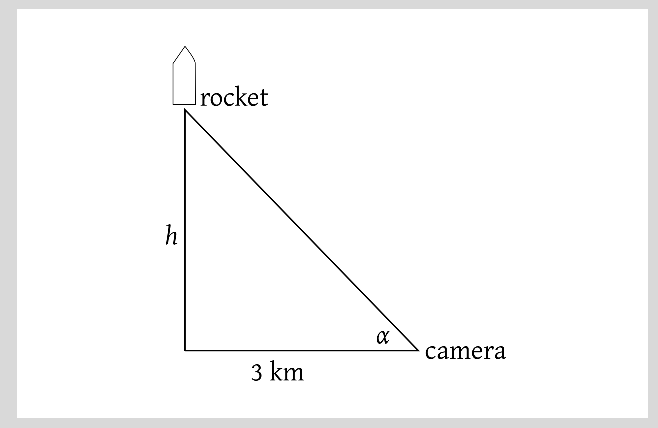
How can we relate the two rates? The quantities α and h are related by a right triangle:



Notice that the measurement h lies on the side opposite α, and we also know that the side adjacent α is constant at 3 km. We can relate the quantities using trigonometry:

$$\tan\alpha = \frac{h}{3}.$$

Third Example

Question: A camera is directed at the launch of a rocket. The camera is 3 km from the rocket. Once the rocket rises 1 km, it has a speed of 1500 km/hr. How fast must the angle of the camera change to remain fixed on the rocket?

The problem asks us to find $d\alpha/dt$, given that $dh/dt = 1500$.

This allows us to differentiate and relate the rates:

$$\tan\alpha = \frac{h}{3}$$

$$\sec^2\alpha \cdot \frac{d\alpha}{dt} = \frac{dh/dt}{3}.$$

(Remember from the Chain Rule that, since we differentiate with respect to t,
the derivatives must be multiplied by dh/dt and $d\alpha/dt$ as appropriate.)

Third Example

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The problem asks us to find $d\alpha/dt$, given that $dh/dt = 1500$.

We can substitute known values at this instant:

$$\tan\alpha = \frac{h}{3}$$

$$\sec^2\alpha \cdot \frac{d\alpha}{dt} = \frac{dh/dt}{3}$$

$$\sec^2\alpha \cdot \frac{d\alpha}{dt} = \frac{1500}{3} = 500.$$

But what is the value of α?

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But what is the value of α?

Again we resort to the original equation. Since $\tan\alpha = h/3$, then $\alpha = \arctan(h/3)$. Thus

$$\alpha = \arctan\frac{1}{3}.$$

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Thus:

$$\tan\alpha = \frac{h}{3}$$

$$\sec^2\alpha \cdot \frac{d\alpha}{dt} = \frac{dh/dt}{3}$$

$$\sec^2\alpha \cdot \frac{d\alpha}{dt} = \frac{1500}{3} = 500$$

$$\sec^2\left(\arctan\frac{1}{3}\right) \cdot \frac{d\alpha}{dt} = 500.$$

Third Example

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The problem asks us to find $d\alpha/dt$, given that $dh/dt = 1500$.

Thus:

$$\begin{aligned}
\tan\alpha &= \frac{h}{3} \\
\sec^2\alpha \cdot \frac{d\alpha}{dt} &= \frac{dh/dt}{3} \\
\sec^2\alpha \cdot \frac{d\alpha}{dt} &= \frac{1500}{3} = 500 \\
\sec^2(\arctan 500) \cdot \frac{d\alpha}{dt} &= 500 \\
\frac{d\alpha}{dt} &= \frac{500}{\sec^2\left(\arctan\frac{1}{3}\right)}.
\end{aligned}$$

Third Example

Question: A camera is directed at the launch of a rocket. The camera is 3 km from the rocket. Once the rocket rises 1 km, it has a speed of 1500 km/hr. How fast must the angle of the camera change to remain fixed on the rocket?

The problem asks us to find $d\alpha/dt$, given that $dh/dt = 1500$.

Thus:

$$\tan\alpha = \frac{h}{3}$$

$$\vdots$$

$$\frac{d\alpha}{dt} = \frac{500}{\sec^2\left(\arctan\frac{1}{3}\right)}.$$

From trigonometry, $\sec^2\left(\arctan\frac{1}{3}\right) = \frac{\sqrt{10}}{3}$. Hence

$$\frac{d\alpha}{dt} = \frac{500}{\frac{\sqrt{10}}{3}} = \frac{1500}{\sqrt{3}} \approx 866 \text{ radians per hour.}$$

Conclusion

- Problems that involve related rates can be solved by taking derivatives.
- The method of solution is to
  - find an equation that relates the quantities;
  - differentiate with respect to time; and
  - solve for the unknown rate.
- Quantities that are constant should not be substituted in before taking the derivative.
- Sometimes you will have to go back and re-examine an equation in order to determine the value of a quantity.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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