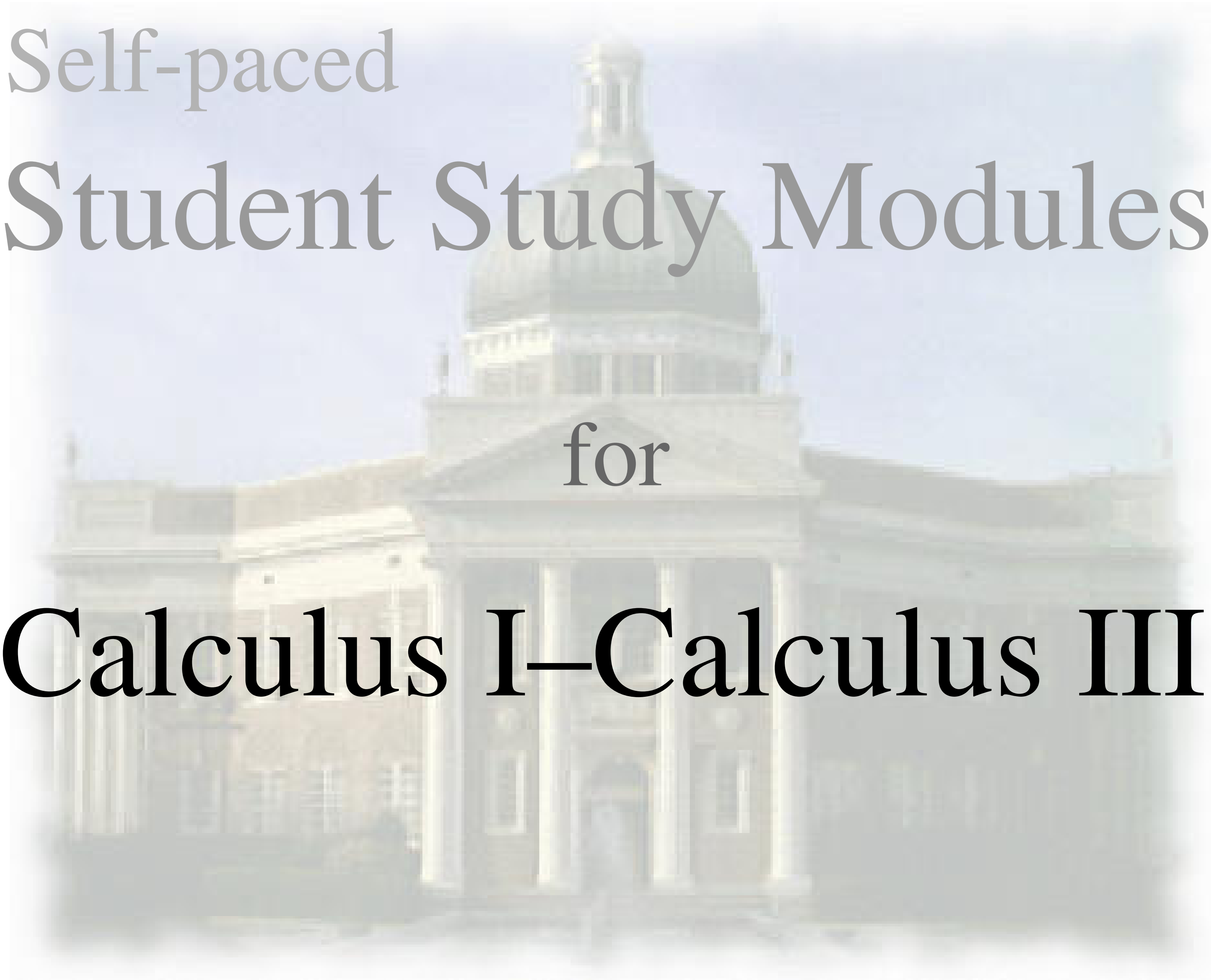

Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Properties of derivatives

As with limits, we do not want to spend a lot of time with the precise definition of the derivative, nor with graphical or numerical approximations. We prefer to develop properties (or “shortcuts”) that allow us to skip the tedious algebra and jump directly into problem solving.

Defining the problem

We would like properties of the derivative that are similar to properties of limits:

- the derivative of a constant;
- the derivative of a constant multiple;
- the derivative of a sum or a difference of functions;
- the derivative of a product of functions; and
- the derivative of a quotient of functions.

SAGE worksheets

You will not need a SAGE worksheet for this module.

The derivative *as a function*

Until now we have mostly considered the derivative *at a point*. Recall the definition:

Definition: *(The derivative at a point)*

The derivative of a function f at $x=a$ is

- $f'(a)=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$, or equivalently
- $f'(a)=\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

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The derivative of a function f at $x = a$ is

- $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, or equivalently
- $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

If f is differentiable at every point in its domain, then we can consider the derivative *as a function* on the same domain as f; that is, we can find a formula that describes $f'(x)$ where x is a *variable* instead of a *constant*. Take the second definition of the derivative at a point, and let the point a vary. What does the definition become?

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Definition: *(The derivative as a function)*
If a function f is differentiable at every point in its domain, then *the derivative of* f is

$$f'(x) = \lim_{x \to a} \frac{f(x+h) - f(x)}{h}.$$

We simply replaced a, which is usually used to represent a constant, with x, which is usually used to represent a variable.

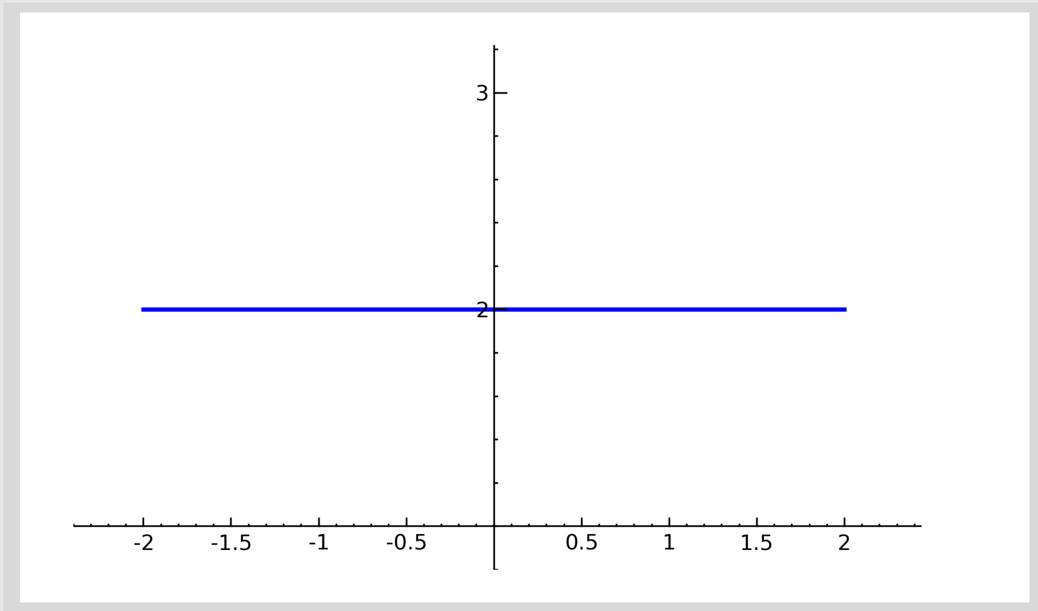
The derivative of a constant

Suppose that $f(x)=c$ where $c\in\mathbb{R}$ is a constant; that is, $f(x)$ has the same value regardless of the value of x. What is the derivative?

The derivative of a constant

Suppose that $f(x)=c$ where $c\in\mathbb{R}$ is a constant; that is, $f(x)$ has the same value regardless of the value of x. What is the derivative?

We can make a guess as to what the derivative *should* be by looking at the graph of a constant, for example $f(x)=2$:



The graph is a horizontal line. Any line tangent to this graph should also be horizontal. The slope of a horizontal line is zero, so we can venture that

$$f'(x)=0.$$

The derivative of a constant

Suppose that $f(x) = c$ where $c \in \mathbb{R}$ is a constant; that is, $f(x)$ has the same value regardless of the value of x. What is the derivative?

The precise definition of the derivative verifies this for us.

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative of a constant

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The precise definition of the derivative verifies this for us.

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{x \to 0} \frac{c - c}{h}.$$

($f(x) = c$ *regardless of* the value of x.)

The derivative of a constant

Suppose that $f(x) = c$ where $c \in \mathbb{R}$ is a constant; that is, $f(x)$ has the same value regardless of the value of x. What is the derivative?

The precise definition of the derivative verifies this for us.

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{x \to 0} \frac{\cancelto{0}{c - c}}{h}.$$

The derivative of a constant

Suppose that $f(x) = c$ where $c \in \mathbb{R}$ is a constant; that is, $f(x)$ has the same value regardless of the value of x. What is the derivative?

The precise definition of the derivative verifies this for us.

$$f'(x) = \lim_{x \to 0} \frac{\cancelto{0}{c - c}}{h}$$
$$= \lim_{x \to 0} 0$$
$$= 0.$$

(The limit of a constant—here, 0—is the constant.)

Properties of derivatives

We have our first property:

Theorem: *(Properties of derivatives)*

Let f be a function.

- If $f(x)$ is constant, then $f'(x) = 0$.

The derivative of a constant multiple

We move next to the derivative of a constant multiple. Rather than make an intuitive argument based on *geometry* here, we can make an intuitive argument based on *properties of limits*. Let f be a function, $c \in \mathbb{R}$ a constant, and $g(x) = cf(x)$. Here g is a *constant multiple* of f.

The derivative of a constant multiple

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In an earlier module, we considered the derivatives of $\sin x$ and $4 \sin x$ at $x = 1$. We found that the first derivative was 0.54, and the second was $2.16 = 4 \cdot 0.54$. This suggests that

$$(g)' = cf'(x).$$

(Here $f(x) = \sin x$ and $c = 4$.)

The derivative of a constant multiple

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By definition, a derivative is a kind of limit, and from the limit of a constant multiple we expect a constant multiple of the limit:

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$$(g)' = cf'(x).$$

A close examination of the matter using the precise definition confirms this:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}.$$

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$$(g)' = cf'(x).$$

A close examination of the matter using the precise definition confirms this.

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}. \end{aligned}$$

(Substituted $g(x) = cf(x)$.)

The derivative of a constant multiple

We move next to the derivative of a constant multiple. Rather than make an intuitive argument based on *geometry* here, we can make an intuitive argument based on *properties of limits*. Let f be a function, $c \in \mathbb{R}$ a constant, and $g(x) = cf(x)$. By definition, a derivative is a kind of limit, and from the limit of a constant multiple we expect a constant multiple of the limit:

$$(g)' = cf'(x).$$

A close examination of the matter using the precise definition confirms this.

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h\to 0} \frac{c\left[f(x+h) - f(x)\right]}{h}. \end{aligned}$$

(Factored a common c.)

The derivative of a constant multiple

We move next to the derivative of a constant multiple. Rather than make an intuitive argument based on *geometry* here, we can make an intuitive argument based on *properties of limits*. Let f be a function, $c \in \mathbb{R}$ a constant, and $g(x) = cf(x)$. By definition, a derivative is a kind of limit, and from the limit of a constant multiple we expect a constant multiple of the limit:

$$(g)' = cf'(x).$$

A close examination of the matter using the precise definition confirms this.

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\ &= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \end{aligned}$$

(The constant multiple of a limit.)

The derivative of a constant multiple

We move next to the derivative of a constant multiple. Rather than make an intuitive argument based on *geometry* here, we can make an intuitive argument based on *properties of limits*. Let f be a function, $c \in \mathbb{R}$ a constant, and $g(x) = cf(x)$. By definition, a derivative is a kind of limit, and from the limit of a constant multiple we expect a constant multiple of the limit:

$$(g)' = cf'(x).$$

A close examination of the matter using the precise definition confirms this.

$$\begin{aligned} g'(x) &= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x). \end{aligned}$$

(The precise definition of $f'(x)$.)

Properties of derivatives

We have our second property:

Theorem: *Properties of derivatives* Let f be a function.

- If $f(x)$ is constant, then the derivative of f is zero.
- If c is a constant and $f'(x)$ exists,
  then the derivative of $cf(x)$ is $cf'(x)$.

The derivative of a sum

Let f, g, h be functions such that $s = f + g$. We want to calculate $s'(x)$; that is, the derivative of $(f+g)(x)$. From the definition of the derivative,

$$s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}.$$

The derivative of a sum

Let f, g, h be functions such that $s = f + g$. We want to calculate $s'(x)$; that is, the derivative of $(f+g)(x)$. From the definition of the derivative,

$$\begin{aligned} s'(x) &= \lim_{h \to 0} \frac{s(x+h) - s(x)}{h} \\ &= \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}. \end{aligned}$$

(Definition of $s(x)$.)

The derivative of a sum

Let f, g, h be functions such that $s = f + g$. We want to calculate $s'(x)$; that is, the derivative of $(f+g)(x)$. From the definition of the derivative,

$$\begin{aligned} s'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}. \end{aligned}$$

(Definition of addition of functions.)

The derivative of a sum

Let f, g, h be functions such that $s = f + g$. We want to calculate $s'(x)$; that is, the derivative of $(f+g)(x)$. From the definition of the derivative,

$$\begin{aligned} s'(x) &= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right). \end{aligned}$$

(Group expressions with f, g.)

The derivative of a sum

Let f, g, h be functions such that $s = f + g$. We want to calculate $s'(x)$; that is, the derivative of $(f+g)(x)$. From the definition of the derivative,

$$\begin{aligned} s'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right) \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}. \end{aligned}$$

(The limit of a sum is the sum of the limits.
We are assuming that the limits being added exist.)

The derivative of a sum

Let f, g, h be functions such that $s = f + g$. We want to calculate $s'(x)$; that is, the derivative of $(f+g)(x)$. From the definition of the derivative,

$$\begin{aligned} s'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= f'(x) + g'(x). \end{aligned}$$

(The precise definition of the derivative.)

Properties of derivatives

We have our third property:

Theorem: *Properties of derivatives* Let f be a function.

- If $f(x)$ is constant, then the derivative of f is zero.
- If c is a constant and $f'(x)$ exists,
then the derivative of $cf(x)$ is $cf'(x)$.
- If the derivatives of $f(x)$ and $g(x)$ exist,
then the derivative of $(f \pm g)(x) = f'(x) \pm g'(x)$.

The derivative of a product

Let f, g, p be functions such that $p = fg$. We want to calculate $p'(x)$; that is, the derivative of $(fg)(x)$. You might be inclinde to think that the derivative of a product is the product of the derivatives; that is,

$$(f(x)g(x))' = f'(x)g'(x).$$

However, *this does not work!* A little pencil grease shows that the derivative of x^2 is $2x$, so the derivative of $x^2 = x \cdot x$ cannot be $x' \cdot x' = 1 \cdot 1 = 1$. We have to consider the definition of the derivative.

The derivative of a product

Let f, g, p be functions such that $p = f\,g$. We want to calculate $p'(x)$; that is, the derivative of $(f\,g)(x)$. From the definition of the derivative,

$$p'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}$$

The derivative of a product

Let f, g, p be functions such that $p = f\,g$. We want to calculate $p'(x)$; that is, the derivative of $(f\,g)(x)$. From the definition of the derivative,

$$\begin{aligned} p'(x) &= \lim_{h \to 0} \frac{s(x+h) - s(x)}{h} \\ &= \lim_{h \to 0} \frac{(f\,g)(x+h) - (f\,g)(x)}{h} \end{aligned}$$

(Definition of $s(x)$.)

The derivative of a product

Let f, g, p be functions such that $p = fg$. We want to calculate $p'(x)$; that is, the derivative of $(fg)(x)$. From the definition of the derivative,

$$
\begin{aligned}
p'(x) &= \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}
\end{aligned}
$$

(Definition of the product of two functions.)

The derivative of a product

Let f, g, p be functions such that $p = f\,g$. We want to calculate $p'(x)$; that is, the derivative of $(f\,g)(x)$. From the definition of the derivative,

$$
\begin{aligned}
p'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) + [f(x+h)g(x) - f(x+h)g(x)] - f(x)g(x)}{h}
\end{aligned}
$$

(Added zero: the terms in the middle cancel.)

The derivative of a product

Let f, g, p be functions such that $p = f\,g$. We want to calculate $p'(x)$; that is, the derivative of $(f\,g)(x)$. From the definition of the derivative,

$$p'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) + [f(x+h)g(x) - f(x+h)g(x)] - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] - g(x)[f(x+h) - f(x)]}{h}$$

(Factored common terms.)

The derivative of a product

Let f, g, p be functions such that $p = fg$. We want to calculate $p'(x)$; that is, the derivative of $(fg)(x)$. From the definition of the derivative,

$$p'(x) = \lim_{h\to 0} \frac{f(x+h)[g(x+h)-g(x)] - g(x)[f(x+h)-f(x)]}{h}$$

$$\stackrel{?}{=} \lim_{h\to 0} \frac{f(x+h)[g(x+h)-g(x)]}{h} - \lim_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h}.$$

(Properties of limits, if these limits exist.)

The derivative of a product

Let f, g, p be functions such that $p = fg$. We want to calculate $p'(x)$; that is, the derivative of $(fg)(x)$. From the definition of the derivative,

$$p'(x) \stackrel{?}{=} \lim_{h\to 0} \frac{f(x+h)[g(x+h)-g(x)]}{h} - \lim_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h}.$$

We need to analyze the limits here. In the second case, $g(x)$ is constant with respect to h, so

$$\lim_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h} = g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = g(x)f'(x).$$

The derivative of a product

Let f, g, p be functions such that $p = f\,g$. We want to calculate $p'(x)$; that is, the derivative of $(f\,g)(x)$. From the definition of the derivative,

$$p'(x) \stackrel{?}{=} \lim_{h\to 0} \frac{f(x+h)(g(x+h)-g(x)]}{h} - \lim_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h}.$$

In the first case, we use the properties of the limit of a product:

$$\lim_{h\to 0} \frac{f(x+h)[g(x+h)-g(x)]}{h} = \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x)\cdot g'(x).$$

The derivative of a product

Let f, g, p be functions such that $p = fg$. We want to calculate $p'(x)$; that is, the derivative of $(fg)(x)$. From the definition of the derivative,

$$\begin{aligned} p'(x) &= \lim_{h\to 0} \frac{f(x+h)(g(x+h)-g(x)]}{h} - \lim_{h\to 0} \frac{g(x)[f(x+h)-f(x)]}{h} \\ &= f(x)g'(x) + f'(x)g(x). \end{aligned}$$

Notice that we did *not* obtain $f'(x)g'(x)$.

Properties of derivatives

We have our fourth property:

Theorem: *Properties of derivatives* Let f be a function.

- If $f(x)$ is constant, then the derivative of f is zero.
- If c is a constant and $f'(x)$ exists,
  then the derivative of $cf(x)$ is $cf'(x)$.
- If the derivatives of $f(x)$ and $g(x)$ exist,
  then the derivative of $(f \pm g)(x)$ is $f'(x) \pm g'(x)$.
- *The product rule*
  If the derivatives of $f(x)$ and $g(x)$ exist,
  then the derivative of $(fg)(x)$ is $f(x)g'(x) + f'(x)g(x)$.

The derivative of a quotient

We will not show all the details of the derivative of a quotient, but the argument proceeds in much the same way as the argument for the derivative of a product. (*Try* it!) Instead, we list all the properties on the next page.

Properties of derivatives

Theorem: *Properties of derivatives* Let f be a function.

- If $f(x)$ is constant, then the derivative of f is zero.
- If c is a constant and $f'(x)$ exists,
  then the derivative of $cf(x)$ is $cf'(x)$.
- If the derivatives of $f(x)$ and $g(x)$ exist,
  then the derivative of $(f \pm g)(x)$ is $f'(x) \pm g'(x)$.
- *(The product rule)*
  If the derivatives of $f(x)$ and $g(x)$ exist,
  then the derivative of $(fg)(x)$ is $f(x)g'(x) + f'(x)g(x)$.
- *(The quotient rule)*
  If the derivatives of $f(x)$ and $g(x)$ exist,
  then the derivative of $(f/g)(x)$ is $\dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$.

Examples

Examples on the following slides use the facts that

the derivative of x^2+1 is $2x$ and the derivative of x^3-1 is $3x^2$.

You should verify these by hand, using the precise definition of the derivative.

Example

Using the constant multiple rule, the derivative of $4(x^2+1)$ is

$$4\cdot\left[\frac{d}{dx}(x^2+1)\right]=4\cdot 2x.$$

(One could simplify this further, but we want you to see the structure.)

You could also verify this by taking the derivative of the product $4(x^2+1)=4x^2+4$, but at this point it would be laborious to use the precise derivative on the product.

Example

Using the product rule, the derivative of $(x^2+1)(x^3-1)$ is

$$(x^2+1)\left[\frac{d}{dx}(x^3-1)\right]+\left[\frac{d}{dx}(x^2+1)\right](x^3-1)=(x^2+1)\cdot 3x^2+2x(x^3-1).$$

(One could simplify this further, but we want you to see the structure.)

You could also verify this by taking the derivative of the product $(x^2+1)(x^3-1) = x^5+x^3-x^2-1$ and comparing it to the simplification of the derivative $(x^2+1)\cdot 3x^2+2x(x^3-1)=5x^4+3x^2-2x$, but at this point it would be laborious to use the precise derivative on the product.

Example

Using the quotient rule, the derivative of $(x^2+1)/(x^3-1)$ is

$$\frac{\left[\frac{d}{dx}(x^2+1)\right](x^3-1)-(x^2+1)\left[\frac{d}{dx}(x^3-1)\right]}{(x^3-1)^2}=\frac{2x(x^3-1)-(x^2+1)\cdot 3x^2}{(x^3-1)^2}.$$

(One could simplify this further, but we want you to see the structure.)

Conclusion

- We determined a definition for the derivative *as a function for arbitrary* x:

> **Definition:** *(The derivative as a function)*
> If a function f is differentiable at every point in its domain, then the derivative of f is
> $$f'(x) = \lim_{x \to a} \frac{f(x+h) - f(x)}{h}.$$

- We identified several properties of the derivative, for
  - a constant;
  - a constant multiple of a function;
  - a sum or difference of two functions;
  - a product of two functions;
  - a quotient of two functions.
- The first three were rather intuitive, based on what we know of limits.
- The last two were counter-intuitive, and required us to play some algebra in the precise definition of the derivative.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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