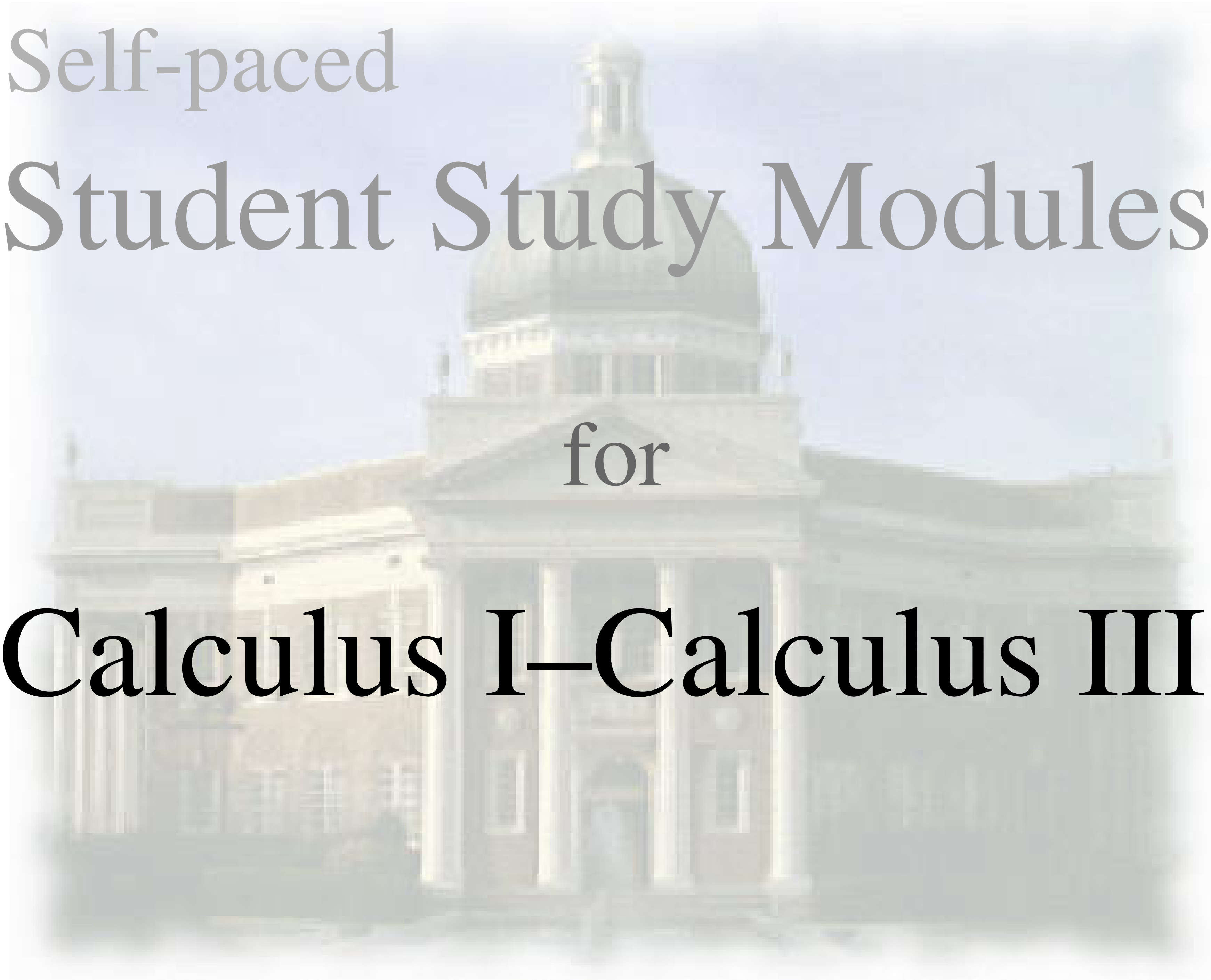


Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages **⟨Next⟩**, backward **⟨Prev⟩**, or view all the slides in this tutorial **⟨Index⟩**.
- The **⟨Back to Calc I⟩** button returns you to the course home page.
- A full symbolic algebra package **⟨Sage⟩** is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text **⟨CalcText⟩** provides a quick search of basic calculus topics.
- You can get help from Google Calculus **⟨GoogleCalc⟩**.
- A monochrome copy of this module is suitable for printing **⟨Print⟩**.

When all else fails, feel free to contact your instructor.

Asymptotes

In this module we give a new definition to an idea you have seen before by defining asymptotes in terms of limits.

Defining the problem

Do the following functions have asymptotes?

- $y = 1/x$
- $y = \ln x$
- $y = (3x^2 + x + 1)/(2x^2 - 2)$

If so, what kind of asymptote, and where? If not, why not?

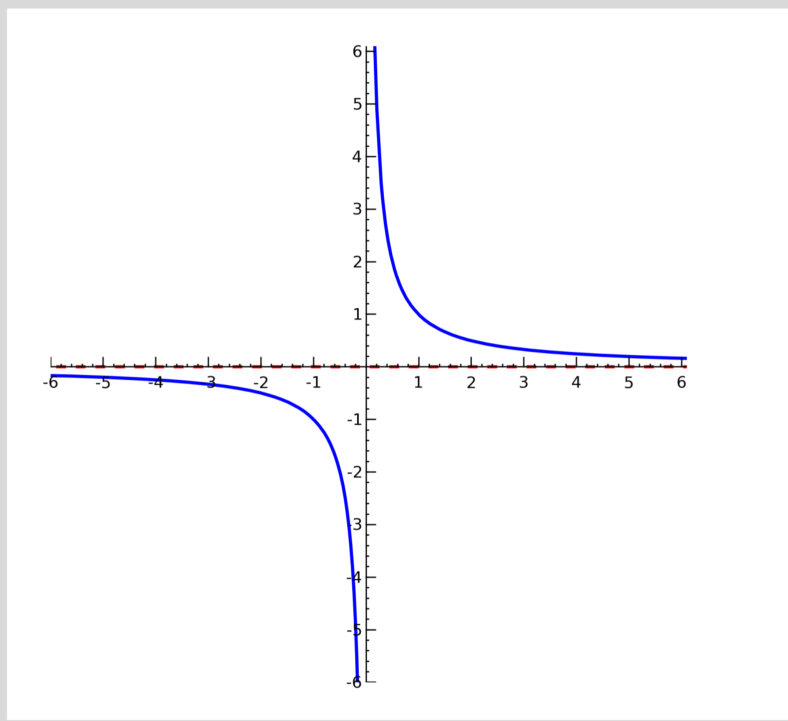
Horizontal asymptotes

In the module on limits at infinity, we noticed that some functions have a finite value for

$$\lim_{x\to\pm\infty} f(x).$$

One example was

$$\lim_{x\to\pm\infty} \frac{1}{x} = 0.$$



Horizontal asymptotes

Definition: *(Horizontal asymptote)*
If $\lim_{x\to\infty} f(x) = L$ where L is a real number, or if $\lim_{x\to-\infty} f(x) = L$ where L is a real number, then the line $y = L$ is a **horizontal asymptote** of $f(x)$.

In the previous example, $y = 0$ is a horizontal asymptote of $1/x$.

Horizontal asymptotes: $\ln x$

On the other hand, what is $\lim_{x\to\infty} \ln x$?

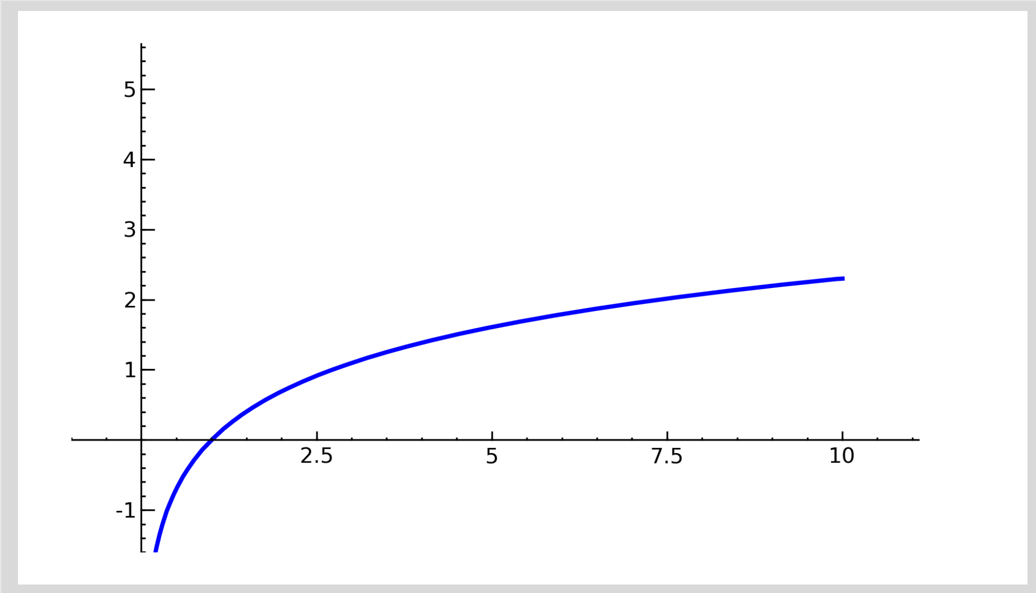


Figure 1: Here $L = 1$ and $\varepsilon = 1$; $x > e^2$ implies that $\ln x > L + \varepsilon$.

Horizontal asymptotes: $\ln x$

On the other hand, what is $\lim_{x\to\infty} \ln x$? Let $\varepsilon = 1$. No matter what value you choose

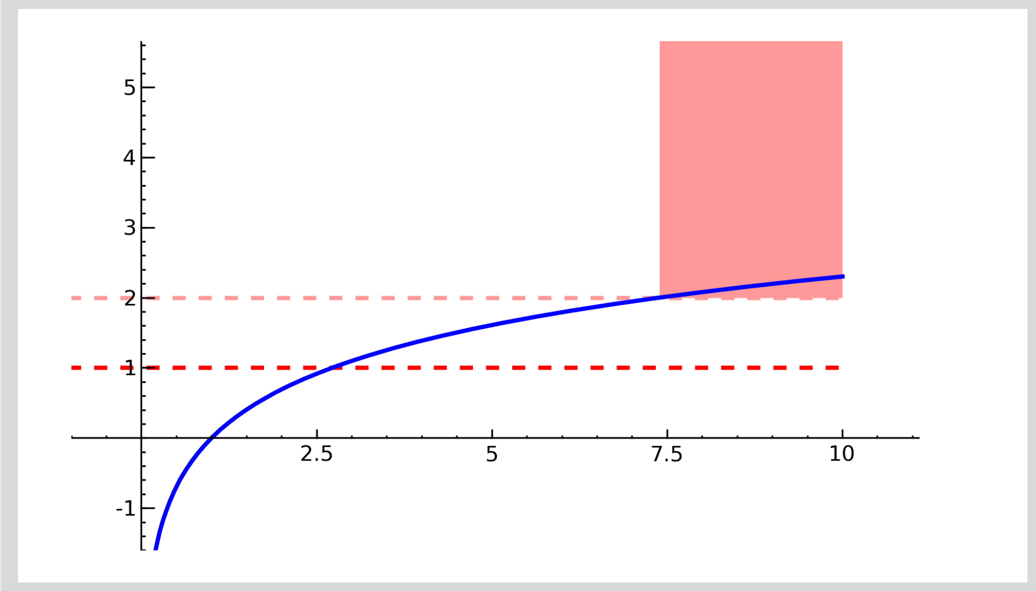


Figure 2: Here $L = 1$ and $\varepsilon = 1$; $x > e^2$ implies that $\ln x > L + \varepsilon$.

for L, eventually $|\ln x - L| > \varepsilon$ for *every* value of x.

Horizontal asymptotes: $\ln x$

On the other hand, what is $\lim_{x\to\infty} \ln x$? Let $\varepsilon = 1$. No matter what value you choose for L, eventually $|\ln x - L| > \varepsilon$ for *every* value of x.

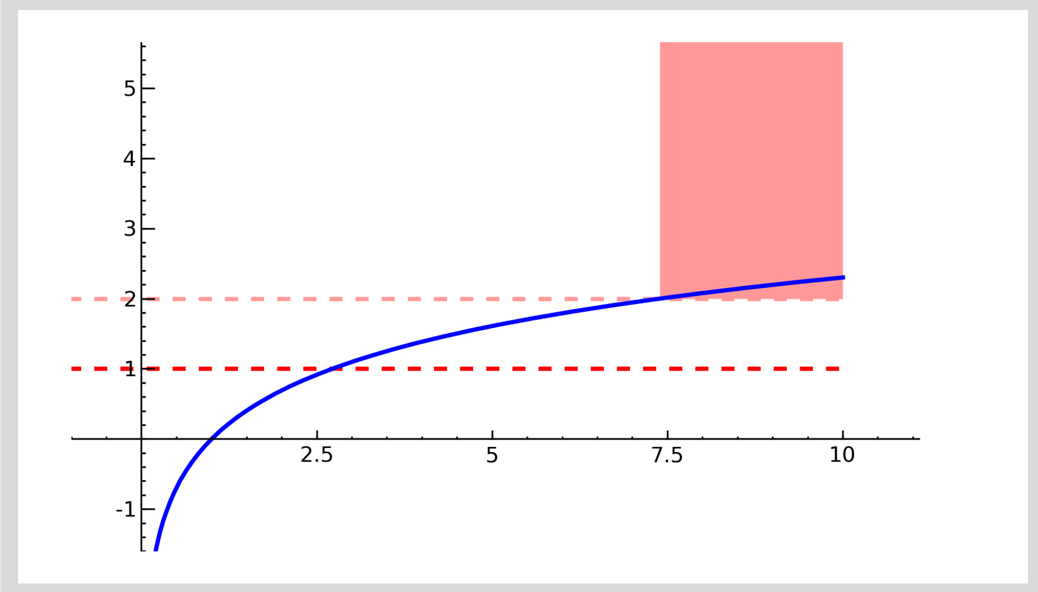


Figure 3: Here $L = 1$ and $\varepsilon = 1$; $x > e^2$ implies that $\ln x > L + \varepsilon$.

Since $\ln x$ is a one-to-one function, if $x > \exp(L+\varepsilon)$ then $|\ln x - L| > \varepsilon$.

Horizontal asymptotes: third example

The third example is

$$f(x) = \frac{3x^2 + x + 1}{2x^2 - 1}.$$

Horizontal asymptotes: third example

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$$f(x) = \frac{3x^2 + x + 1}{2x^2 - 1}.$$

To find the limit, we rewrite this expression, multiplying numerator and denominator by $1/x^2$ to obtain

$$f(x) = \frac{3x^2 + x + 1}{2x^2 - 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{3 + \frac{1}{x} + \frac{1}{x^2}}{2 - \frac{1}{x^2}}.$$

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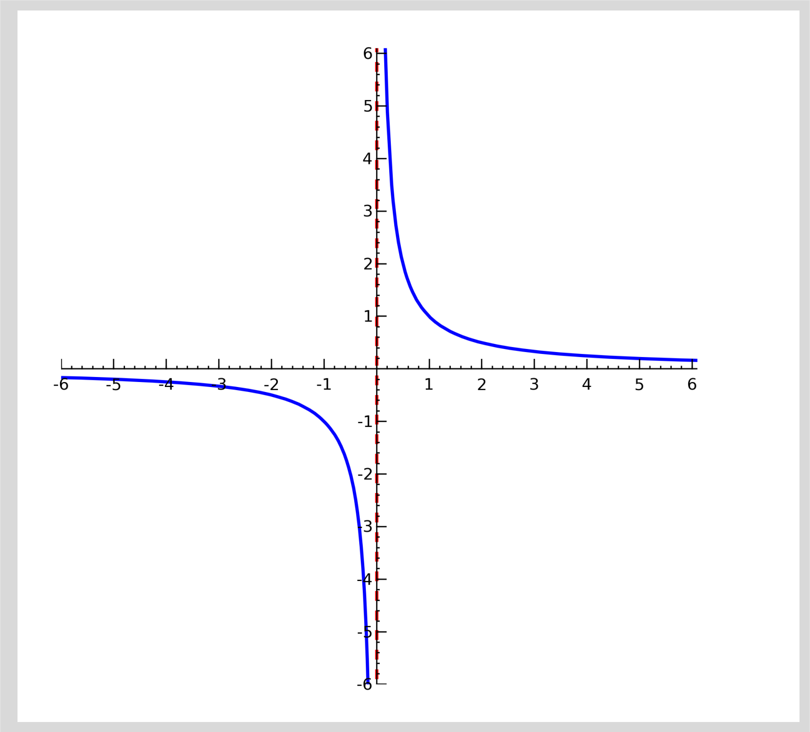
This allows us to say that

$$\lim_{x\to\infty} f(x)=\lim_{x\to 0}\frac{3+\cancelto{0}{\frac{1}{x}}+\cancelto{0}{\frac{1}{x^2}}}{2-\cancelto{0}{\frac{1}{x^2}}}=\frac{3}{2}.$$

Vertical asymptotes

Now we move to vertical asymptotes. The graph of $1/x$ has a vertical asymptote at $x = 0$.



How can we describe this using limits?

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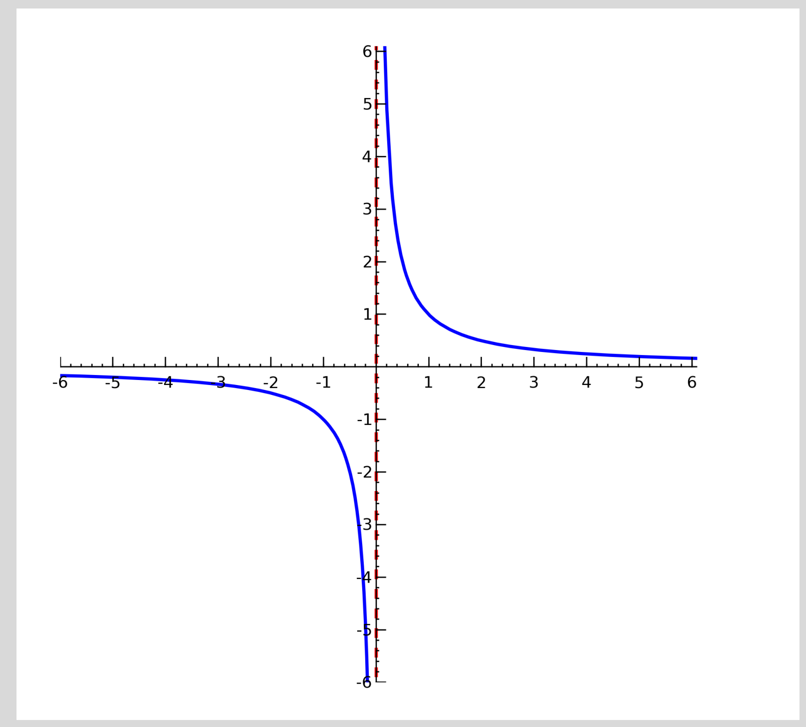


How can we describe this using limits? Notice that the y-values increase dramatically when the x-values approach 0.

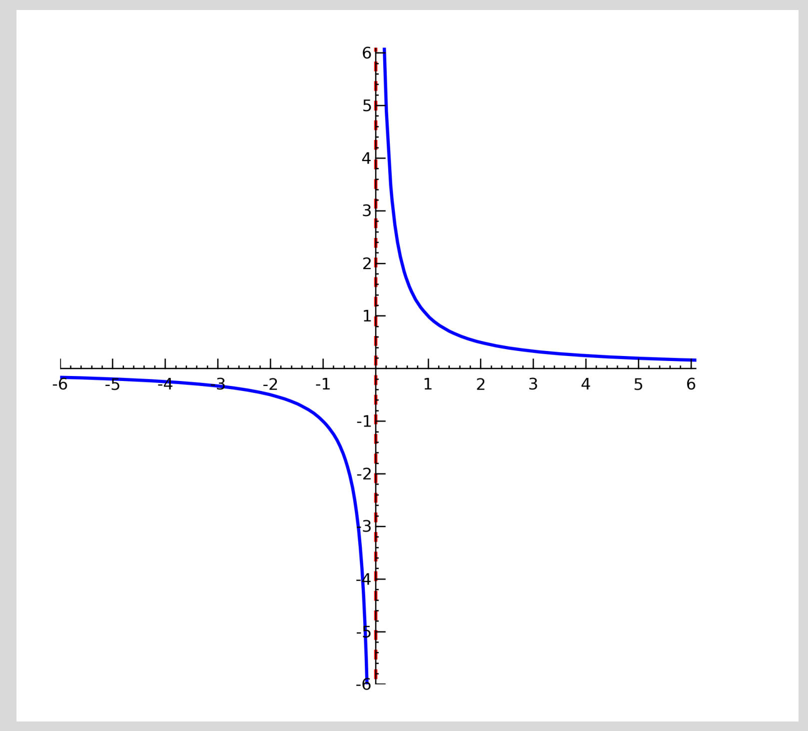
Vertical asymptotes: definition

That suggests the following definition.

Definition: *(Vertical asymptotes)*

A **vertical asymptote** occurs at $x = a$ when

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty.$$



Limit *of* infinity

Here we encounter a new idea:

$$\lim_{x \to a} f(x) = \pm\infty.$$

Intuitively it makes sense, but we'd best give a precise definition for it.

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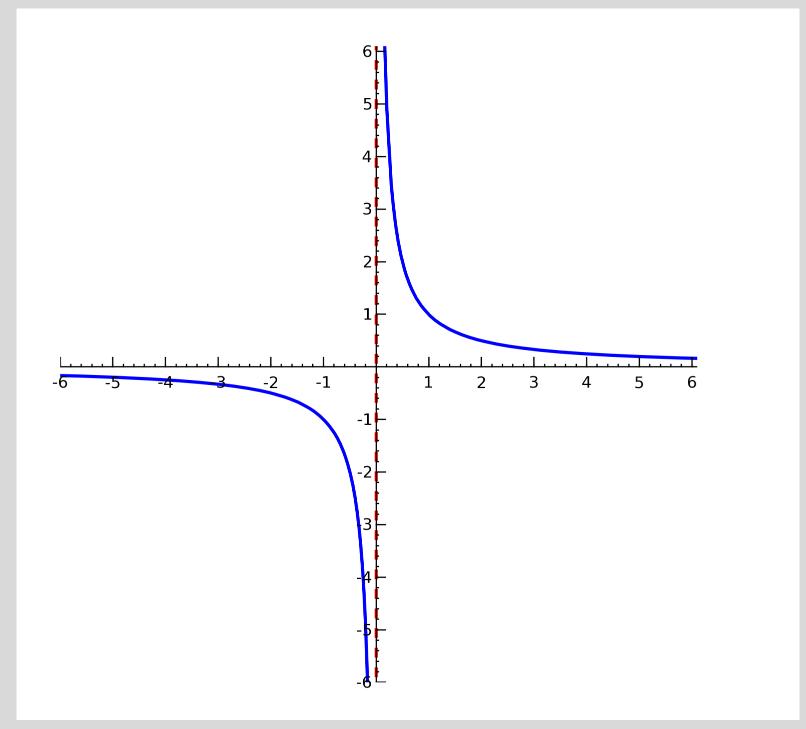
We want to capture the notion that

- when the x values draw arbitrarily close to a,
- the y values increase (or decrease) without bound.



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Go back and review the precise definitions of (a) a finite limit, and (b) a limit *at* infinity. See if you can combine those two ideas into the definition of a limit *of* infinity.

A limit *of* infinity?

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We want to capture the notion that

- when the x values draw arbitrarily close to a,
- the y values increase (or decrease) without bound.

Definition: *(Limit of ∞)*

$\lim_{x\to a} = \infty$ if

- for all $N \in \mathbb{R}$,
- there exists $\delta > 0$ such that
  - for all x satisfying $|x - a| < \delta$ (except maybe $x = a$), then
  - $f(x) > N$ for all $f(x)$ (except maybe $f(a)$).

Study the definition to make sure you understand how it captures the notion.

A limit of $-\infty$?

We can build a similar definition for a limit of $-\infty$.

Definition: *(Limit of $-\infty$)*

$\lim_{x\to a} = -\infty$ if

- for all $N \in \mathbb{R}$,
- there exists $\delta > 0$ such that
  - for all x satisfying $|x - a| < \delta$ (except maybe $x = a$), then
  - $f(x) < N$ for all $f(x)$ (except maybe $f(a)$).

Detecting vertical asymptotes: division by zero?

Definition: *(Vertical asymptotes)*
A **vertical asymptote** occurs at $x=a$ when

$$\lim_{x\to a^+} f(x)=\pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x)=\pm\infty.$$

In the case of $1/x$, we can notice this from division by zero. If a function has the form $f(x)/g(x)$ such that $x\to a$ implies that

- $g(x)$ approaches zero, and
- $f(x)$ approaches a nonzero constant,

then the function has a vertical asymptote at $x=a$. We won't prove this observation, but you should think about why it's true.

Theorem: If $f(x)\to 1/0$ when $x\to a$ from the left or from the right, then $\lim_{x\to a} f(x)=\infty$ or $\lim_{x\to a} f(x)=-\infty$.

Detecting vertical asymptotes: division by zero?

Theorem: If $f(x) \to 1/0$ when $x \to a$ from the left or from the right, then $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$.

It's important that the numerator approaches a *nonzero* constant. If both the numerator and the denominator approach zero, then there may be a hole instead of an asymptote, as with the example

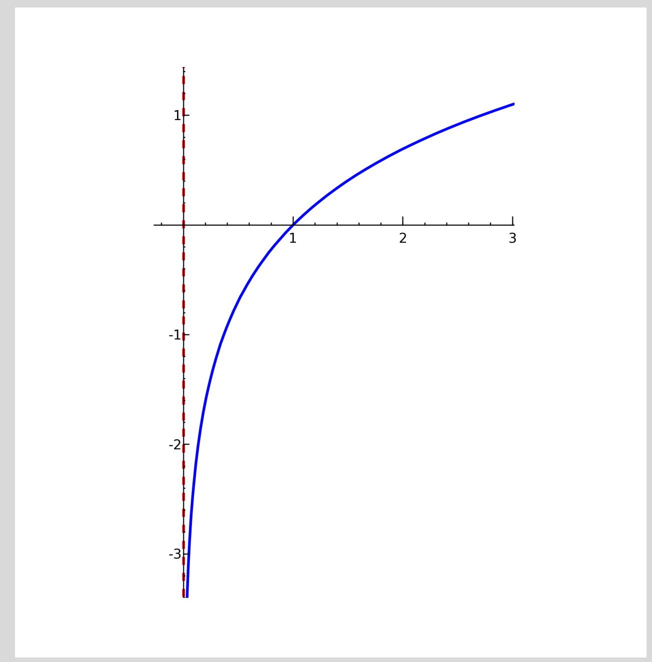
$$\frac{3x+1}{x}$$

in the module on limits at infinity.

Detecting vertical asymptotes

Not all functions with vertical asymptotes fit this mold. In other cases, you must perform a careful analysis before you can decide whether there is a vertical asymptote. This is the case with $\ln x$: graphical and numerical checks suggest that it has a vertical asymptote at $x = 0$. Since $\ln x$ is undefined for $x < 0$, we need check only that $\lim_{x\to 0^+} \ln x = -\infty$.

x	1	0.1	0.01	0.001	0.0001
$\ln x$	0	-2.3026	-4.6052	-6.9078	-9.2103



Detecting vertical asymptotes

If we want to use the precise definition and show that $\ln x < N$ for any real number N once we are close enough to zero, the choice $x = e^{-N}$ suffices.

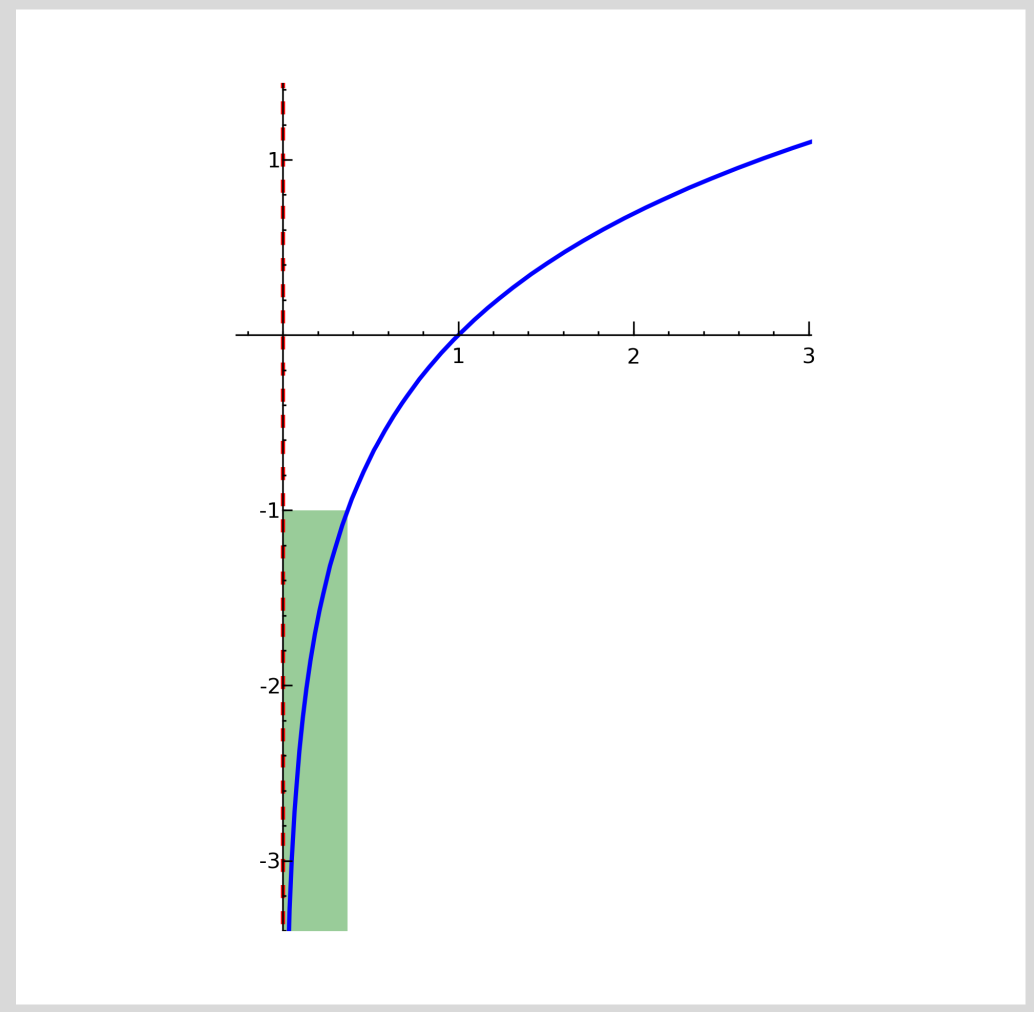


Figure 4: If $x < e^{-1}$, then $\ln x < -1$.

Other asymptotes

Some functions have asymptotes that are neither horizontal nor vertical. These can be oblique asymptotes, or nonlinear asymptotes. In each case, the function has the form $a(x)+z(x)$ where as $x \to \infty$, $z(x) \to 0$.

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For example, if

$$f(x) = \frac{x^2+1}{x+1},$$

polynomial division shows that

$$f(x) = x-1+\frac{2}{x+1}.$$

Here $a(x) = x-1$ and $z(x) = \frac{2}{x+1}$. As $x \to \pm\infty$, $\frac{2}{x+1} \to 0$ so $x-1$ is an oblique asymptote.

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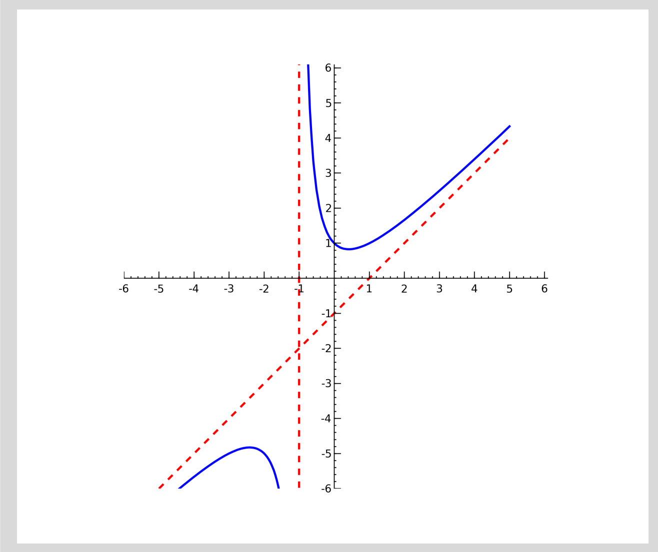

Figure 5: Oblique asymptote $x-1$.

Other asymptotes

Some functions have asymptotes that are neither horizontal nor vertical. These can be oblique asymptotes, or nonlinear asymptotes. In each case, the function has the form $a(x)+z(x)$ where as $x \to \infty$, $z(x) \to 0$.

For a nonlinear example, take

$$f(x) = \ln x + \frac{1}{x}.$$

Here $a(x) = \ln x$ and $z(x) = \frac{1}{x}$. As $x \to \pm\infty$, $\frac{1}{x} \to 0$ so $\ln x$ is a nonlinear asymptote.

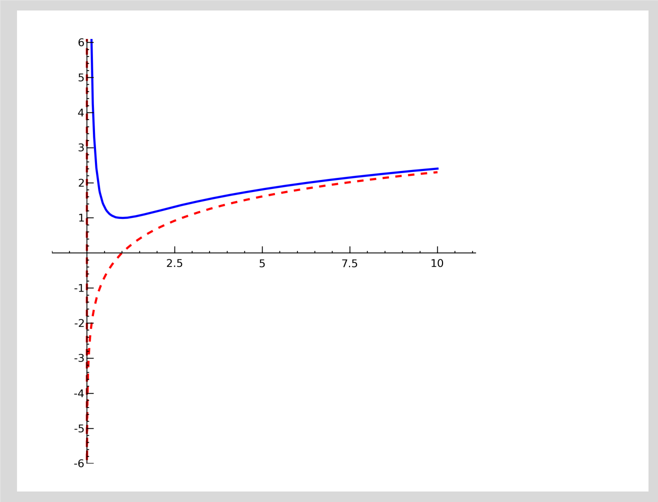

Figure 6: The nonlinear asymptote $\ln x$.

A complete example

Find the asymptotes of

$$\frac{3x^2+x+1}{2x^2-2}.$$

A complete example

Find the asymptotes of

$$\frac{3x^2+x+1}{2x^2-2}.$$

We start by looking for vertical asymptotes. The only place this can occur is when the denominator is zero (all the other x-values are defined); that is,

$$\begin{aligned} 2x^2-2&=0 \\ x^2&=1 \\ x&=\pm 1. \end{aligned}$$

A complete example

Find the asymptotes of
$$\frac{3x^2+x+1}{2x^2-2}.$$

It has vertical asymptotes at $x=\pm 1$.

Now we consider horizontal, oblique, or nonlinear asymptotes.

$$\lim_{x\to\pm\infty}\frac{3x^2+x+1}{2x^2-2}=\lim_{x\to\pm\infty}\frac{\left(3x^2+x+1\right)\cdot\frac{1}{x^2}}{\left(2x^2-2\right)\cdot\frac{1}{x^2}}$$

(Multiply top and bottom by the hightest power of x.)

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$$=\lim_{x\to\pm\infty}\frac{3+\frac{1}{x}+\frac{1}{x^2}}{2-\frac{2}{x^2}}$$

(Distributed $1/x^2$.)

A complete example

Find the asymptotes of

$$\frac{3x^2+x+1}{2x^2-2}.$$

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$$=\lim_{x\to\infty}\frac{3+\cancelto{0}{\frac{1}{x}}+\cancelto{0}{\frac{1}{x^2}}}{2-\cancelto{0}{\frac{2}{x^2}}}$$

(Using $\lim_{x\to\pm\infty}1/x=0$ and limit properties...)

A complete example

Find the asymptotes of

$$\frac{3x^2+x+1}{2x^2-2}.$$

It has vertical asymptotes at $x=\pm 1$.

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$$\lim_{x\to\pm\infty}\frac{3x^2+x+1}{2x^2-2}=\lim_{x\to\infty}\frac{3+\cancelto{0}{\frac{1}{x}}+\cancelto{0}{\frac{1}{x^2}}}{2-\cancelto{0}{\frac{2}{x^2}}}$$

$$=\frac{3}{2}.$$

A complete example

Find the asymptotes of

$$\frac{3x^2+x+1}{2x^2-2}.$$

It has vertical asymptotes at $x=\pm 1$. It has a horizontal asymptote at $y=3/2$.

There are no oblique or non-linear asymptotes; long division of polynomials gives us

$$\frac{3x^2+x+1}{2x^2-2}=\frac{3}{2}+\frac{x+4}{2x^2-2},$$

and as $x\to\pm\infty$ this approaches $3/2$, which corresponds to the horizontal asymptote.

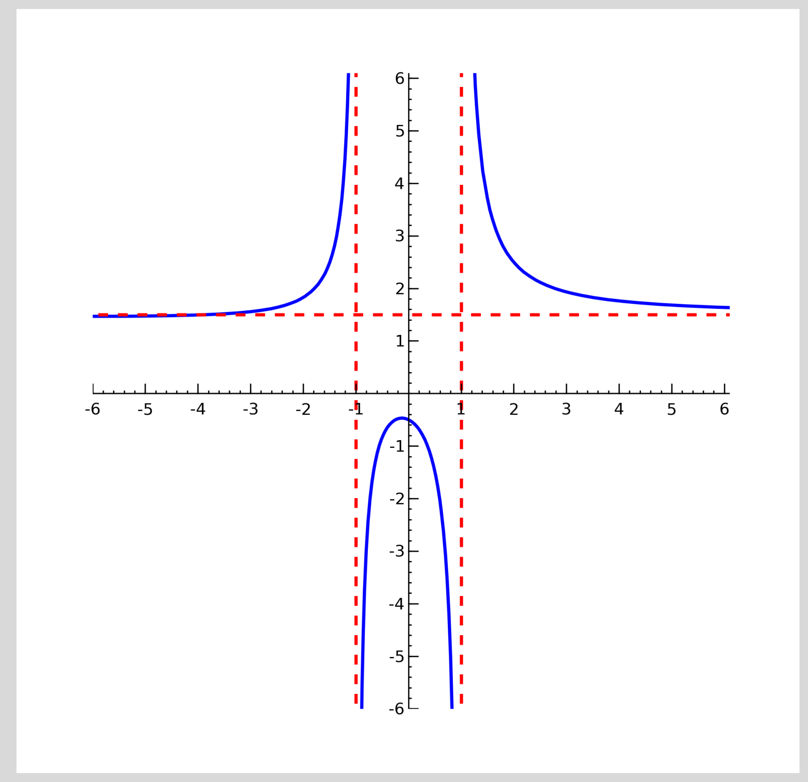
A complete example

Find the asymptotes of

$$\frac{3x^2+x+1}{2x^2-2}.$$

It has vertical asymptotes at $x=\pm 1$. It has a horizontal asymptote at $y=3/2$.

Combining this information and plotting a few points, we end up with this graph:



(The graph crosses $y=-3/2$ on the left, but still approaches it eventually.)

Conclusion

We considered different ways to find the asymptotes of a function $g(x)$. All of these come from limits:

- A horizontal asymptote exists at $x = a$ if $\lim_{x\to a} g(x) = \pm\infty$. This is true even if the limit is only from one side.
- A vertical asymptote exists at $y = L$ if $\lim_{x\to\infty} g(x) = L$. This is true for $x \to -\infty$ as well.
- An oblique or nonlinear asymptote $f(x)$ exists if $g(x) = a(x) + z(x)$ where $z \to 0$ when $x \to 0$.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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