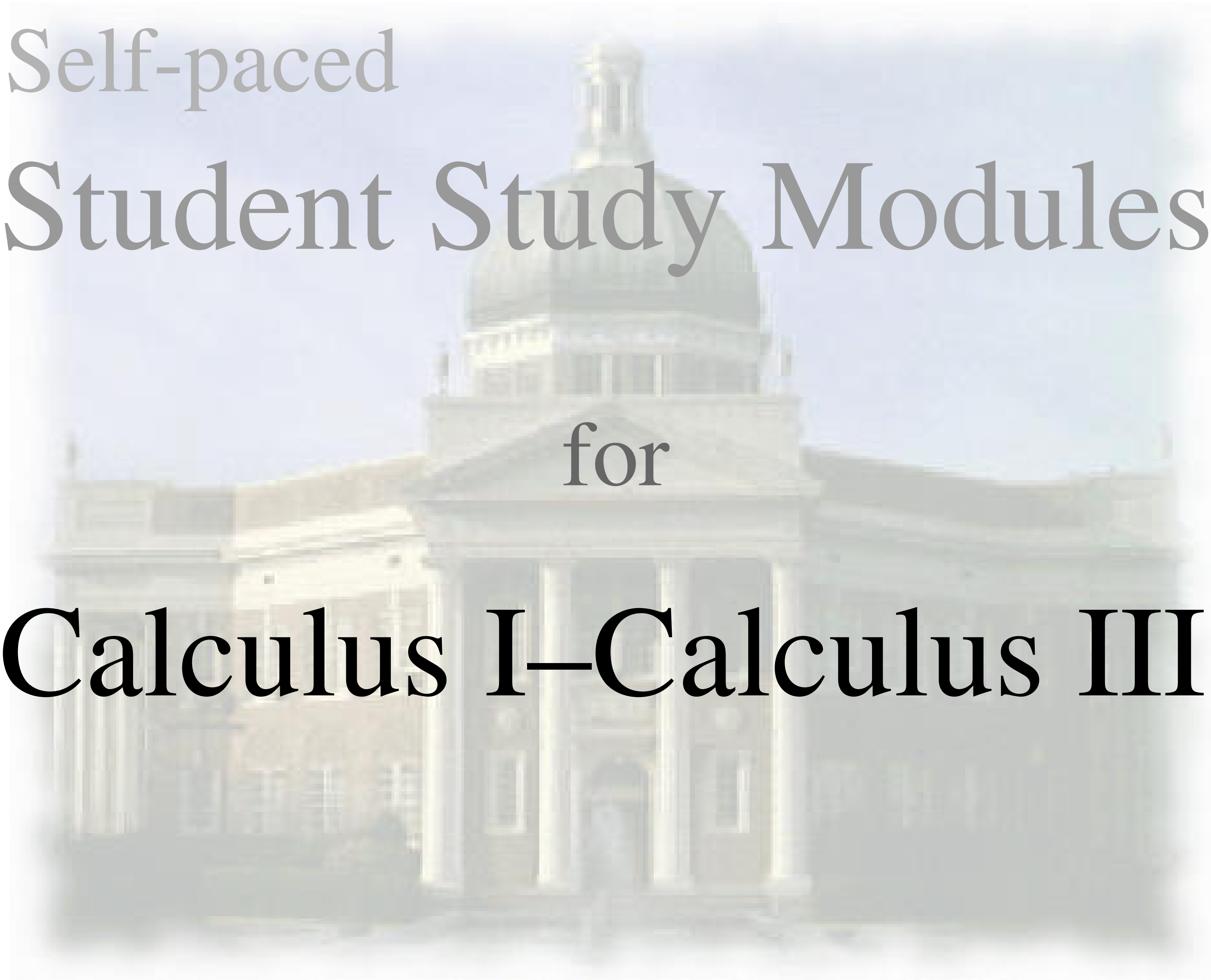

Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages **⟨Next⟩**, backward **⟨Prev⟩**, or view all the slides in this tutorial **⟨Index⟩**.
- The **⟨Back to Calc I⟩** button returns you to the course home page.
- A full symbolic algebra package **⟨Sage⟩** is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text **⟨CalcText⟩** provides a quick search of basic calculus topics.
- You can get help from Google Calculus **⟨GoogleCalc⟩**.
- A monochrome copy of this module is suitable for printing **⟨Print⟩**.

When all else fails, feel free to contact your instructor.

Overview

An important application of Calculus is to problems of *optimization*. In an optimization problem, someone wants to *minimize* or *maximize* the value of a variable. This can be applied to

- economics, where businesses wish to maximize profits;
- aeronautics, where engineers wish to minimize drag; or
- biology, where animals instinctively minimize distance traveled.

Sample problem

Question:

- Find the dimensions of a cylindrical can that minimize surface area. Assume that the thickness of the can is the same everywhere and that the can will hold 1 L of fluid.
- Find the dimensions of a cylindrical can that minimize surface area. Assume that the top of the can must be twice as thick as the other sides and that the can will hold 1 L of fluid.

SAGE worksheets

You will not need a SAGE worksheet for this module.

Strategy

Optimization is related to the topic of finding *extrema*. Thus the procedure is exactly the same, with the exception that in optimization the function that you want to minimize or maximize is not usually known beforehand.

- Determine the function $f(x)$ that needs to be minimized or maximized.
- Find all places where $f'(x)=0$ or $f'(x)$ does not exist. These x values are called critical points.
- Find the y values at each critical points *and* at the endpoints.
- The smallest y value found is the minimum, and the largest y value found is the maximum.

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It asks us to minimize the surface area of a cylinder. The relevant equation is

$$A = 2\pi r h + 2\pi r^2.$$

(The area of the cylinder is the sum of the areas of the top, the bottom, and the side. The top and the bottom are circles of radius r, so their area is πr^2 each. We can cut and unfold the side to obtain a rectangle, whose width is h and whose length is the circumference of the circles on top and bottom: $2\pi r$. Add the areas and you have

$$\begin{aligned} A &= (\text{side}) + (\text{top}) + (\text{bottom}) \\ &= 2\pi r h + \pi r^2 + \pi r^2. \end{aligned}$$

Simplifying this gives the equation above.)

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> **Question:** Find the dimensions of a cylindrical can that minimize surface area. Assume that the thickness of the can is the same everywhere and that the can will hold 1 L of fluid.

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Here A is the function that depends on the variables r and h. We encounter a difficulty here: do we differentiate A with respect to r or to h? Before we can take the derivative, we need to simplify the equation. *How?*

Other information

In this problem, you also need to use the assumption that the can will hold 1 L of fluid. This is information about the *volume* of the can.

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The volume of a cylinder is

$$V = \pi r^2 h.$$

We can substitute 1 L for V and solve for h, obtaining

$$h = \frac{1}{\pi r^2}.$$

Other information: WATCH OUT!

In this problem, you also need to use the assumption that the can will hold 1 L of fluid. This is information about the *volume* of the can.

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We have actually made a subtle mistake here. We want to compute a measure of *distance* (h) and V is a measure of *volume*. Before using it in the problem, we have to convert 1 L to 1000 cm^3.

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Since $h = 1000/(\pi r^2)$, we can rewrite A as

$$A = 2\pi r \left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2.$$

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$$A' = -\frac{2000}{r^2} + 4\pi r.$$

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$$0 = -2000 + 4\pi r^3$$

(Multiplied both sides by r^2.)

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$$r = \sqrt[3]{\frac{500}{\pi}}.$$

(Solved for r.)

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Here A is the function that depends on the variables r and h.

So $A' = 0$ when $r = \sqrt[3]{500/\pi}$. The problems asks for the dimensions, so we also need to find h:

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\sqrt[3]{250{,}000\pi}}.$$

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So $A' = 0$ when $r = \sqrt[3]{500/\pi}$ and $h = 1000/\sqrt[3]{250,000\pi}$. Rounded to two digits, the dimensions are

$$r \approx 5.42 \text{ cm and } h \approx 10.84 \text{ cm}.$$

(This value corresponds to real-world measurements of cans that hold 1 L of oil. Why would someone want to minimize the surface area of a can?)

Second example

The second example is,

Question: Find the dimensions of a cylindrical can that minimize surface area. Assume that the top of the can must be twice as thick as the other sides and that the can will hold 1 L of fluid.

The difference in this problem is that the top is twice as thick as the other sides, giving

$$A = 3\pi r^2 + 2\pi r h.$$

(The area of the top circle is doubled, so the total area is $2\pi r + 2\pi r^2 + \pi r^2$.)

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You should find that

$$r = \sqrt[3]{\frac{1000}{3\pi}} \approx 4.73 \text{ cm and}$$
$$h = \sqrt[3]{\frac{9000}{\pi}} \approx 14.20 \text{ cm.}$$

Draw a picture of a can with these dimensions. Can you think of any cans in the supermarket that look like that picture?

Conclusion

- We showed how derivatives can be used to solve for maximum and minimum values of a function.
- The method is essentially that of optimization:

  - Determine the function $f(x)$ that needs to be minimized or maximized.
  - Find all places where $f'(x)=0$ or $f'(x)$ does not exist. These x values are called critical points.
  - Find the y values at each critical points *and* at the endpoints.
  - The smallest y value found is the minimum, and the largest y value found is the maximum.

- You will often have to use additional information to rewrite f as a function in only *one* variable.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the **⟨Print⟩** icon, and then saving or printing the pdf file.

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