Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

The Chain Rule

In a previous module we discussed properties of derivatives, and in subsequent modules we used them to describe shortcuts for computing the derivatives of polynomials and exponential functions.

In this module we turn to another property of derivatives, the Chain Rule.

Defining the problem

Question: How can we compute the derivatives of the following functions *quickly*?

- $\sin(x^2-1)$
- $(x^2-x+1)^{100}$

SAGE worksheets

You do not need a SAGE worksheet for this module.

The Chain Rule

We start by stating *the Chain Rule*:

Theorem: Let g and u be functions such that u is differentiable on its domain and g is differentiable on the range of u. If $f = g \circ u$ then

$$f'(x) = g'(u) \cdot u'(x).$$

Alternatively, in Leibniz notation

$$\frac{df}{dx} = \frac{dg}{du} \cdot \frac{du}{dx}.$$

The Leibniz notation does *not* prove the theorem, although it suggests it. We will not prove this theorem (your text or your instructor can provide a proof) but we will review quite a few examples.

A note on the Chain Rule

Look back at the Chain Rule and observe the structure of f: it is a *composition of functions*. The Chain Rule's strength is that it helps us compute derivatives of functions with this structure. The examples given at the beginning of the module are of this kind:

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  - $g(u) = \sin u$, and
  - $u(x) = x^2 - 1$.

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- $\sin(x^2-1)=(g\circ u)(x)$ where
  - $g(u)=\sin u$, and
  - $u(x)=x^2-1$.
- $(x^2-x+1)^{100}=(g\circ u)(x)$ where
  - $g(u)=u^{100}$, and
  - $u(x)=x^2-x+1$.

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  - $g(u)=u^{100}$, and
  - $u(x)=x^2-x+1$.

Compositions of functions often look as if they have an "inside" and an "outside"; here u is the "inside" function and g is the "outside" function.

First Example

In the first example, we have $f(x)=\sin(x^2-1)=(g\circ u)(x)$ where

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Application of the Chain Rule gives

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(Definition of the Chain Rule.)

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$$= \cos u \cdot (2x).$$

(Substituted the derivatives.)

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$$f'(x) = \cos u \cdot (2x)$$
$$= 2x \cos\left(x^2 - 1\right).$$

(Substituted for u.)

Second Example

In the second example, we have $f(x) = \left(x^2 - x + 1\right)^{100} = (g \circ u)(x)$ where

- $g(u) = u^{100}$, and
- $u(x) = x^2 - x + 1$.

Second Example

In the second example, we have $f(x) = \left(x^2 - x + 1\right)^{100} = (g \circ u)(x)$ where

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Application of the Chain Rule gives

$$\begin{aligned} f'(x) &= 100u^{99}\cdot(2x) \\ &= 200\left(x^2-x+1\right)^{99}. \end{aligned}$$

(Substituted for u.)

Third Example

Let's try a third example. Let $f(x)=\tan\left((x-3)^3+(x-3)^2-1\right)$. Start by identifying f and u in this function.

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We have

- $g(u)=\tan u$, and
- $u(x)=(x^2-3)^3+(x^2-3)^2-1$.

(It is possible to obtain a different answer. Check with your instructor if you think you have a different answer that is correct.)

Third Example

Let's try a third example. Let $f(x) = \tan\left((x-3)^3 + (x-3)^2 - 1\right)$. Start by identifying f and u in this function.

We have

- $g(u) = \tan u$, and
- $u(x) = (x^2-3)^3 + (x^2-3)^2 - 1$.

Notice that $u(x)$ is also a composition of functions! Before we can compute $f'(x)$ we will have to compute $u'(x)$.

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We have

- $g(u) = \tan u$, and
- $u(x) = (x^2-3)^3 + (x^2-3)^2 - 1$.

Put $u(x) = (g_1 \circ u_1)(x)$ where

- $g_1(u_1) = u_1^3 + u_1^2 - 1$, and
- $u_1(x) = x^2 - 3$.

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Put $u(x) = (g_1 \circ u_1)(x)$ where

- $g_1(u_1) = u_1^3 + u_1^2 - 1$, and
- $u_1(x) = x^2 - 3$.

Then

$$u'(x) = g_1'(u_1) \cdot u_1'(x).$$

(Definition of the Chain Rule.)

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Then

$$\begin{aligned} u'(x) &= g_1'(u_1) \cdot u_1'(x) \\ &= (3u_1^2 + 2u_1) \cdot 2x. \end{aligned}$$

(Substituted the derivatives.)

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Then

$$\begin{aligned} u'(x) &= (3u_1^2 + 2u_1)\cdot 2x \\ &= 2x\left[3(x^2-3)^2 + 2(x^2-3)\right]. \end{aligned}$$

(Substituted for u_1.)

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Before we can compute $f'(x)$ we will have to compute

$$u'(x) = 2x\left[3(x^2-3)^2 + 2(x^2-3)\right].$$

We finally put them together:

$$f'(x) = g'(u)\cdot u'(x).$$

(The Chain Rule.)

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Before we can compute $f'(x)$ we will have to compute

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We finally put them together:

$$\begin{aligned} f'(x) &= g'(u)\cdot u'(x) \\ &= \sec^2 u \cdot 2x\left[3(x^2-3)^2 + 2(x^2-3)\right]. \end{aligned}$$

(Substituted for g' and u'.)

Module: The Chain Rule 7

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We finally put them together:

$$\begin{aligned} f'(x) &= \sec^2 u \cdot 2x\left[3(x^2-3)^2 + 2(x^2-3)\right] \\ &= \sec^2\left[(x^2-3)^3 + (x^2-3)^2 - 1\right]\cdot 2x\left[3(x^2-3)^2 + 2(x^2-3)\right]. \end{aligned}$$

(Substituted for u.)

Numerical Example

In the fourth example, the formulas for f, g, and u are not known, but the following values *are* known.

x	$g(x)$	$g'(x)$	$u(x)$	$u'(x)$
-2	-2	10	5	-10
-1	5	2	-3	-5
0	7	0	0	-4
1	6	-3	-2	0
2	0	-2	-1	3

If $f(x) = (g \circ u)(x)$, what is $f'(2)$?

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If $f(x) = (g \circ u)(x)$, what is $f'(2)$?

Again, we use the Chain Rule. We have

$$f'(x) = g'(u) \cdot u'(x).$$

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If $f(x) = (g \circ u)(x)$, what is $f'(2)$?

We want values of $f'(x)$ when $x = 2$, or

$$f'(x) = g'(u) \cdot u'(x)$$
$$f'(2) = g'(u(2)) \cdot u'(2).$$

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If $f(x) = (g \circ u)(x)$, what is $f'(2)$?

We can determine those values from the table:

$$f'(x) = g'(u) \cdot u'(x)$$
$$f'(2) = g'(u(2)) \cdot u'(2)$$
$$= g'(-1) \cdot 3 = 2 \cdot 3 = 6.$$

So $f'(2) = 6$.

Conclusion

- The Chain Rule is a shortcut for functions derived by composition:

Theorem: Let g and u be functions such that u is differentiable on its domain and g is differentiable on the range of u. If $f = g \circ u$ then

$$f'(x) = g'(u) \cdot u'(x).$$

Alternatively, in Leibniz notation

$$\frac{df}{dx} = \frac{dg}{du} \cdot \frac{du}{dx}.$$

- To use the Chain Rule, we must identify an "*inside*" function and an "*outside*" function.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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