Self-paced

Student Study Modules

for

Calculus I–Calculus III

Instructions

This tutorial session is color coded to assist you in finding information on the page. It is online, but it is important to take notes and to work some of the examples on paper.

- You can move forward through the pages ⟨**Next**⟩, backward ⟨**Prev**⟩, or view all the slides in this tutorial ⟨**Index**⟩.
- The ⟨**Back to Calc I**⟩ button returns you to the course home page.
- A full symbolic algebra package ⟨**Sage**⟩ is accessible online. You can download and install it on your own computer, without a web app, by visiting `www.sagemath.org`.
- An online calculus text ⟨**CalcText**⟩ provides a quick search of basic calculus topics.
- You can get help from Google Calculus ⟨**GoogleCalc**⟩.
- A monochrome copy of this module is suitable for printing ⟨**Print**⟩.

When all else fails, feel free to contact your instructor.

Overview

In general, it is not sufficient to examine the graph of a function to decide on the limit, because any practical graph lacks information that may be essential to the final result. This requires us to move from the *intuitive* notion of a limit (which until now has been numerical and/or geometric) to the *precise* notion of a limit (which is logical and algebraic).

A Sample problem

Let

- $f(x) = \dfrac{x^2 - x}{x}$, and
- $w(x) = \sin\left(\dfrac{1}{x}\right)$.

Our goal is to find a *precise* way to explain our intuitions that

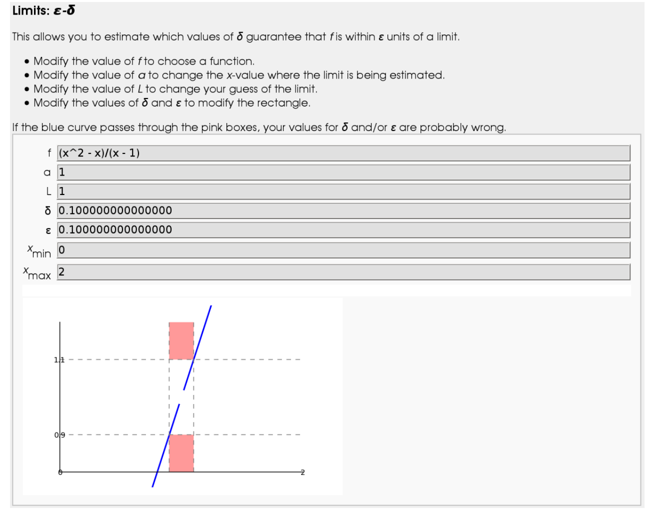
$$\lim_{x\to 0} f(x) = -1 \text{ but } \lim_{x\to 0} w(x) \text{ does not exist.}$$

SAGE worksheets

In this lab you will need the SAGElet "Limits: ϵ-δ". Open it in SAGE.

Setup Limits: ϵ-δ

First we will set up the applet. When you first start it, you should see something like this:



Setup Limits: ϵ-δ

We will set up the applet to work with $f(x)$ first. Make the following assignments:

- $f \to (x^2 - x)/x$
- $a \to 0$
- $L \to -1$
- $\delta \to 0.1$
- $\epsilon \to 0.1$
- $x_{min} \to -1$
- $x_{max} \to 1$

The precise definition of a limit

We want to express the idea that as x draws near to a, the y values $f(x)$ draw so near to L that you cannot find any *meaningful* distance between them. Stated precisely,

Definition: $\lim_{x\to a} f(x) = L$ if

- for any measurable distance ϵ,
- we can find a neighborhood around $x = a$
- such that for any x value in that neighborhood (except maybe $x = a$),
- the distance between the y value of f and L is less than ϵ.

The precise definition of a limit

We want to express the idea that as x draws near to a, the y values $f(x)$ draw so near to L that you cannot find any *meaningful* distance between them. Written more precisely,

Definition: $\lim_{x\to a} f(x) = L$ if

- for all $\epsilon > 0$,
  ("any measurable distance ϵ")
- there exists $\delta > 0$
  ("a neighborhood around $x = a$")
- such that for all $x \in (a-\delta, a+\delta)$ (except maybe $x = a$),
  ("any x value in that neighborhood")
- $|f(x) - L| < \epsilon$.
  ("the distance between the y values is less than ϵ")

Switch back and forth between these two slides to see how the words translate into symbols.

The precise definition of a limit

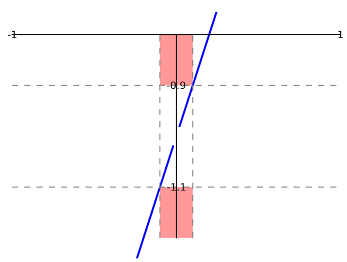
We want to express the idea that as x draws near to a, the y values $f(x)$ draw so near to L that you cannot find any *meaningful* distance between them. Rewritten slightly, we have

Definition: *(Precise definition of the limit)*
$\lim_{x\to a} f(x) = L$ if

- for all $\epsilon > 0$,
- there exists $\delta > 0$
- such that for all x satisfying $|x - a| < \delta$ (except maybe $x = a$),
- $|f(x) - L| < \epsilon$.

Example with $f(x)$

We illustrate this precise definition of a limit using $f(x)$.



Currently, SAGE should show the output above. Right now we set the precision to $\epsilon = 0.1$. The goal is that $|f(x) - L| < \epsilon$, or that *the distance between the y values of f and L is smaller than ϵ.*

Geometrically, you see that the graph of $f(x)$ does not stray into the pink regions of the plane. How does this reflect the definition?

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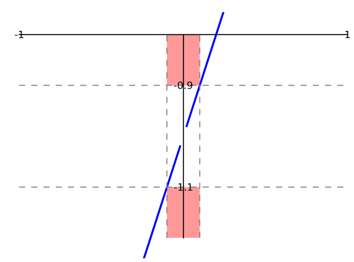

Geometrically, you see that the graph of $f(x)$ does not stray into the pink regions of the plane. How does this reflect the definition?

The distance between the y values of points in the pink regions and L is larger than ϵ. This would violate the desired outcome of a limit: that y values grow very close to L.

What about smaller values of ϵ, say $\epsilon = 0.01$? Change the value of ϵ in the SAGElet.

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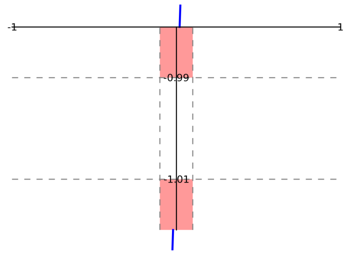

Geometrically, you see that the graph of $f(x)$ does not stray into the pink regions of the plane. How does this reflect the definition?

When we make ϵ smaller, we have a problem: the blue curve of f now passes in the pink regions of the plane. For these points, $|f(x)-L|>\epsilon$.

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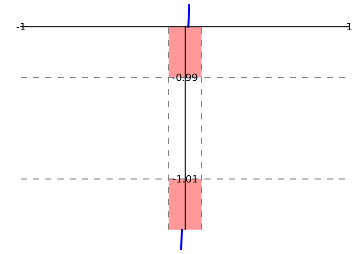


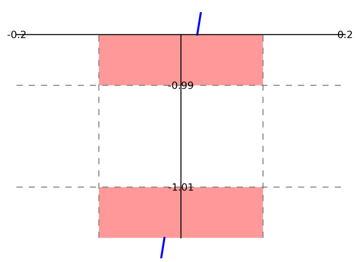
Geometrically, you see that the graph of $f(x)$ does not stray into the pink regions of the plane. How does this reflect the definition?

When we make ϵ smaller, we have a problem: the blue curve of f now passes in the pink regions of the plane. For these points, $|f(x)-L|>\epsilon$.

However, the phrasing of the definition is that "there exists δ". Maybe we can find another value of δ so that the curve no longer enters the pink region. To get an idea, let's zoom in on the graph. Change the maximum and minimum x values so that we zoom into the region of the plane $x\in[-0.2,0.2]$.

Example with $f(x)$

We illustrate this precise definition of a limit using $f(x)$.

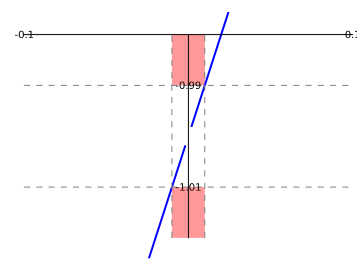


Geometrically, you see that the graph of $f(x)$ does not stray into the pink regions of the plane. How does this reflect the definition?

It looks like we could make it work by choosing a smaller value of δ. Let's try $\delta=0.01$.

Example with $f(x)$

We illustrate this precise definition of a limit using $f(x)$.



Geometrically, you see that the graph of $f(x)$ does not stray into the pink regions of the plane. How does this reflect the definition?

In this case it worked out. Notice that we have $\delta=\epsilon$.

Proving it

We can show this rigorously by falling back on the definition. We remind you that the definition is

- For all $\epsilon > 0$,
- there exists $\delta > 0$
- such that for all x satisfying $|x-a| < \delta$ (except maybe $x=a$),
- $|f(x)-L| < \epsilon$.

We want to rewrite

$$|f(x)-L| < \epsilon$$

so that it resembles $|x-a| <$ something, or $|x-0| <$ something. Then we could just pick $\delta =$ something.

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- such that for all x satisfying $|x-a| < \delta$ (except maybe $x=a$),
- $|f(x)-L| < \epsilon$.

That is,

$$|f(x)-L| < \epsilon$$
$$\left|\frac{x^2-x}{x} - (-1)\right| < \epsilon.$$

(Substituted f and L.)

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That is,

$$|f(x)-L| < \epsilon$$
$$\left|\frac{x^2-x}{x} - (-1)\right| < \epsilon$$
$$\left|\frac{x(x-1)}{x} + 1\right| < \epsilon.$$

(Factored the numerator.)

Proving it

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- such that for all x satisfying $|x-a| < \delta$ (except maybe $x=a$),
- $|f(x)-L| < \epsilon$.

That is,

$$|f(x)-L| < \epsilon$$
$$\left|\frac{x(x-1)}{x} + 1\right| < \epsilon$$
$$|(x-1)+1| < \epsilon, \text{ or } |x-0| < \epsilon.$$

(Division by x when $x \neq 0$.)

So we want $|x| < \epsilon$.

Proving it

We can show this rigorously by falling back on the definition. We remind you that the definition is

- For all $\epsilon > 0$,
- there exists $\delta > 0$
- such that for all x satisfying $|x-a| < \delta$ (except maybe $x = a$),
- $|f(x) - L| < \epsilon$.

What value of δ should we choose? We want

$$|x-0| < \delta \implies |x-0| < \epsilon.$$

Choosing $\delta = \epsilon$ makes it easy.

Proving it

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What value of δ should we choose? We want

$$|x-0| < \delta \implies |x-0| < \epsilon.$$

Choosing $\delta = \epsilon$ makes it easy.

We saw this in the SAGElet: when $\epsilon = .1$ we chose $\delta = .1$, and when $\epsilon = .01$ we chose $\delta = .01$.

Smaller δ works, too!

Of course you could have chosen a δ that was even *smaller* than the one required. Try changing δ to .005 in the SAGE worksheet, and see if the curve passes through the pink region.

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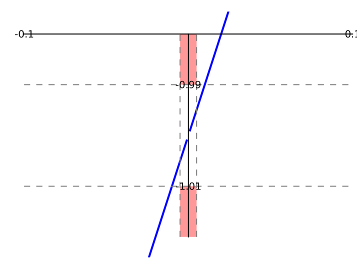
You should see something like this:



So smaller values of delta work just as well.

General functions

It is not easy to determine a correct value of δ for every function. Even “simple” functions require care.

Suppose $f(x) = x^2$. How can we show that $\lim_{x \to 1} f(x) = 1$?

Falling back on the definition, let ϵ be an arbitrary distance. We want

$$|x - a| = |x - 1| < \delta \qquad \Longrightarrow \qquad |f(x) - L| = |x^2 - 1| < \epsilon.$$

Assignment: the precise notion of a limit 8

General functions

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$$|x - a| = |x - 1| < \delta \qquad \Longrightarrow \qquad |f(x) - L| = |x^2 - 1| < \epsilon.$$

We’ll try again to rewrite the second inequality in the form

$$|x - a| = |x - 1| < \text{something}.$$

Then we can set $\delta = \text{something}$ and satisfy the definition.

Assignment: the precise notion of a limit 8

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Falling back on the definition, let ϵ be an arbitrary distance. We want

$$|x - 1| < \delta \qquad \Longrightarrow \qquad |x^2 - 1| < \epsilon.$$

The second inequality can be rewritten as

$$-\epsilon < x^2 - 1 < \epsilon.$$

(Property of the absolute-value inequality <.)

Assignment: the precise notion of a limit 8

General functions

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The second inequality can be rewritten as

$$-\epsilon < x^2 - 1 < \epsilon$$

$$1 - \epsilon < x^2 < 1 + \epsilon.$$

(Isolated x^2.)

General functions

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Falling back on the definition, let ϵ be an arbitrary distance. We want

$$|x-1|<\delta \qquad \Longrightarrow \qquad |x^2-1|<\epsilon.$$

The second inequality can be rewritten as

$$1-\epsilon < x^2 < 1+\epsilon$$
$$\sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}.$$

(Took square root of both sides.
In a "small enough" neighborhood of $x=a=1$,
x is positive, so we don't need $\pm\sqrt{\ }$.)

General functions

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$$|x-1|<\delta \qquad \Longrightarrow \qquad |x^2-1|<\epsilon.$$

The second inequality can be rewritten as

$$\sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}$$
$$\sqrt{1-\epsilon}-1 < x-1 < \sqrt{1+\epsilon}-1.$$

(Subtracted 1 from all sides. Why?
To find a value for δ we need the form $|x-1|$.)

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$$|x-1|<\delta \qquad \Longrightarrow \qquad |x^2-1|<\epsilon.$$

The second inequality can be rewritten as

$$\sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}$$
$$\sqrt{1-\epsilon}-1 < x-1 < \sqrt{1+\epsilon}-1.$$

Which value gives a *smaller* neighborhood around $x=a=1$? Some algebra shows that $\sqrt{1+\epsilon}-1$ (we skip that step here).

General functions

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$$|x-1|<\delta \qquad \Longrightarrow \qquad |x^2-1|<\epsilon.$$

The second inequality can be rewritten as

$$\sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}$$
$$\sqrt{1-\epsilon}-1 < x-1 < \sqrt{1+\epsilon}-1.$$

Try this in the SAGE applet with a few values of ϵ, and verify that $\delta=\sqrt{1+\epsilon}-1$ keeps the blue function out of the pink regions.

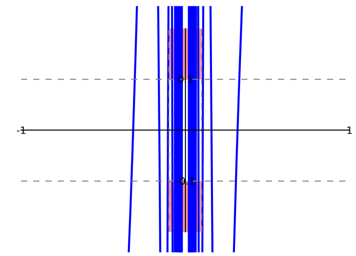
What about $w(x)$?

Now consider $w(x) = \sin(1/x)$. To help see why this has no limit at $x = 0$, make the following changes to the SAGE applet:

- $f \to sin(1/x)$
- $a \to 0$
- $L \to 0$ (we'll start with a guess that the limit is zero)
- $\delta \to 0.1$
- $\epsilon \to 0.1$
- $x_{min} \to -1$
- $x_{max} \to 1$

What about $w(x)$?

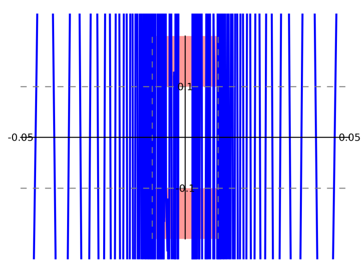
Your initial graph should look like this:



Notice the intense "wiggling" as $x \to 0$. The blue curve passes through the pink regions many, many times, so this value of δ is too large for $\epsilon = .1$; let's try a smaller value of δ, say $\delta = .01$. Be sure to zoom in the x values as well to $[-.05, .05]$.

What about $w(x)$?

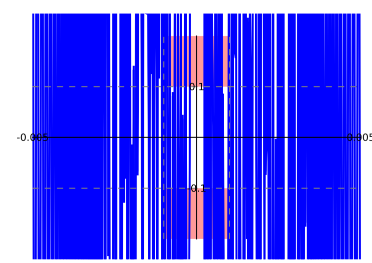
Now you should see this (or something similar):



The wiggling has not ceased to take us into the pink zone, so δ is still too large for this value of ϵ. Let's try one more smaller value of δ.

What about $w(x)$?

Now you should see this (or something similar):



No wonder mathematicians call this the *Infinite Wiggle*.

What is going on?

No matter how small we made δ, the graph of $w(x)$ only wiggled all the more intensely. We need to convince ourselves of this.

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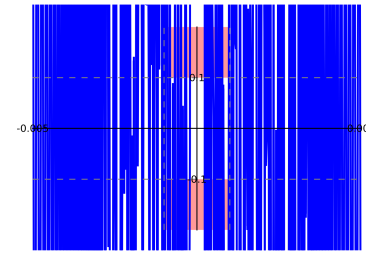
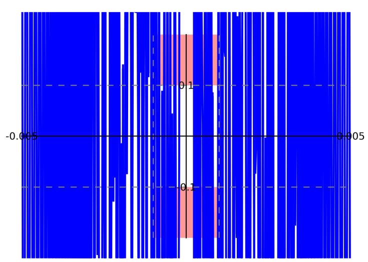


Notice that the curve touches $y = 1$ many, many times. What x values give $\sin(x) = 1$?

What is going on?

No matter how small we made δ, the graph of $w(x)$ only wiggled all the more intensely. We would like to explain this as much as possible.



Notice that the curve touches $y = 1$ many, many times. What x values give $\sin(x) = 1$?
You should know that $x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{5\pi}{2}, -\frac{5\pi}{2}, \ldots$ produce $\sin(x) = 1$.

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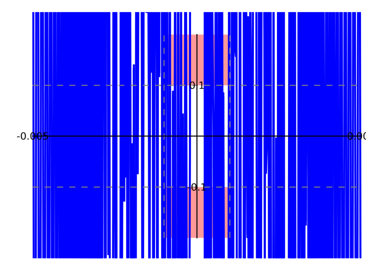

Notice that the curve touches $y = 1$ many, many times. What x values give $\sin(x) = 1$?
You should know that $x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{5\pi}{2}, -\frac{5\pi}{2}, \ldots$ produce $\sin(x) = 1$.
We are dealing with the function $\sin(1/x)$, so look at the reciprocal of these numbers: $x = \frac{2}{\pi}, -\frac{2}{\pi}, \frac{2}{5\pi}, -\frac{2}{5\pi}, \ldots$ produce $\sin(1/x) = 1$.

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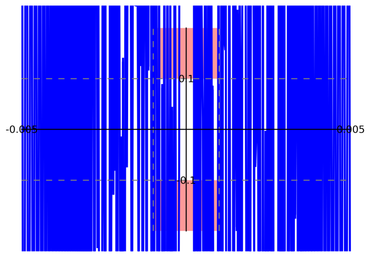

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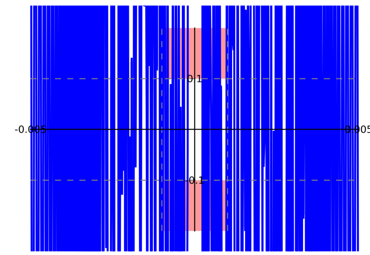
These values approach zero: $\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$. The pattern goes on forever, *all* their y values are 1, and the limit of these numbers is 0.

Assignment: the precise notion of a limit 11

What is going on?

No matter how small we made δ, the graph of $w(x)$ only wiggled all the more intensely. We need to convince ourselves of this.



The upshot?

- Regardless of the value of δ
- we can find x
- such that $|x - 0| < \delta$
- but $|f(x) - 0| = 1 > 0.1 = \epsilon$!

Thus...

We conclude that

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

does not exist.

Conclusion

- You have seen the precise definition of a limit:

  Definition: $\lim_{x \to a} f(x) = L$ if
  - for all $\epsilon > 0$,
  - there exists $\delta > 0$
  - such that for all x satisfying $|x - a| < \delta$ (except maybe $x = a$)
  - $|f(x) - L| < \epsilon$.
- You have had an introduction to finding δ algebraically. It takes a lot of practice to grow accustomed to it.
- Why does this give us the precise definition of the limit?
  - No matter how close (ϵ) we check near $y = L$,
  - we can find a neighborhood (δ) of $x = a$
  - such that the distance between L and $f(x)$ is less than ϵ ($|f(x) - L| < \epsilon$).
  - Since ϵ is *any* measurable distance, the limit of $f(x)$ is L.
- You can use the SAGElet "Limits: ϵ-δ" to illustrate this, both when the limit exists and when it does not.

End of Module

Please review your work, select another module, or select an option from the top menu.

You may also obtain a black and white condensed version of this tutorial by clicking the ⟨**Print**⟩ icon, and then saving or printing the pdf file.

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The University of
Southern Mississippi