Sage-Combinat and Combinatorial Species

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Sage-Combinat

My Involvement with Sage

- I first heard about Sage at Sage Days 3 in February 2007.
- I had already written a fair amount of Python code for the combinatorics of the representation theory of the symmetric group.
- I found that MuPAD-Combinat already had much of the functionality that I was interested in.

MuPAD-Combinat

- Open source library for algebraic combinatorics built on top of MuPAD.
- Its main purpose is to provide an extensible toolbox for computer exploration, and foster code sharing between researchers in this area.
- Started in 2000 by Nicolas Thiéry and Florent Hivert.
- Many contributors, 25+ research articles, over 120,000 lines of code, rich category hierarchy, and more.
- Can be found at http://mupad-combinat.sourceforge.net/

Sage Days 7

- Nicolas Thiéry, Anne Schilling, and Jason Bandlow from MuPAD-Combinat attended and worked with others on implementing crystals in Sage.
- The project showed that the possiblity of porting MuPAD-Combinat to Sage was feasible.
- Since Sage Days 7, there has been continued interaction between the two projects resulting a fair amount of code being produced for Sage.

Development Model

Requirements

1. Make it easy for developers to share and work collaboratively on unfinshed code.
2. Make it easy to move finished code into the Sage library.

Development Model

Mercurial Queue Patch Repository

- We develop patches that apply on top of the Sage library.
- It is easy to apply, unapply, and edit any of the patches.
- It is easy to migrate patches across Sage releases.
- When a patch or series of patches are ready, they are sent off to the Sage trac server.
- When they are merged, we mark them as merged so they can be "removed" from our repository when the next Sage is released.

Combinatorial Species

What are species?

Species include trees, graphs, functions, relations, permutations, cycles, lists, lattices, posets, automata, finite geometries, finite groups, ...

The theory of species is a theoretical framework concerned with enumerating these types of objects. Google has very generously funded me to work on implementing combinatorial species in Sage this summer.

What are species?

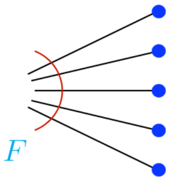
More specifically...

Let $\mathbb{B}$ be the category of finite sets with bijections. A *species* is simply a functor

$$F : \mathbb{B} \to \mathbb{B}.$$

More specifically...

- For every finite set A, we get a finite set $F[A]$ whose elements are said to be the *structures* of F on the underlying set A.

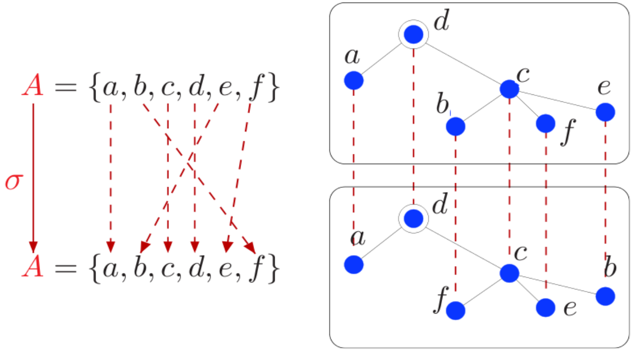


More specifically...

- For each bijection $\phi : A \to B$, we have a bijection

$$F[\phi] : F[A] \to F[B]$$

which is called the transport of F-structures along ϕ.



More specifically...

- F is *functorial*, which means that
  1. $F[\mathrm{Id}_A] = \mathrm{Id}_{F[A]}$
  2. $F[\psi\phi] = F[\psi]F[\phi]$.

Generating Series

To each species F, we associate the *(exponential) generating series*

$$F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

where f_n is the number of elements of $F[A]$ for any A with n elements.

Isomorphism Type Generating Series

We say that two elements $s, t \in F[A]$ are isomorphic if there exists a bijection $\phi : A \rightarrow A$ such that

$$F[\phi](s) = t.$$

The isomorphism type generating series of F is defined to be

$$\tilde{F}(x) = \sum_{n \geq 0} \tilde{f}_n x^n$$

where $\tilde{f}_n$ is the number of non-isomorphic elements of $F[A]$ for any A with n elements.

Example: Partition Species

We define the species of partitions P by letting $P[A]$ be all set partitions of A. Then,

$$P(x) = \sum_{n \geq 0} B_n \frac{x^n}{n!}$$

and

$$\tilde{P}(x) = \sum_{n \geq 0} p_n x^n$$

where B_n are the Bell numbers and p_n is the number of integer partitions of n.

Operations on species

Addition

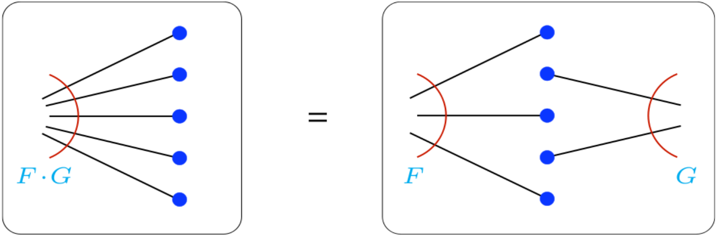
$$(F + G)[A] = F[A] + G[A]$$

The sum on the right side corresponds to a disjoint union.

Operations on species

Multiplication

$$(F \cdot G)[A] = \sum_{B+C=A} F[B] \times G[C]$$

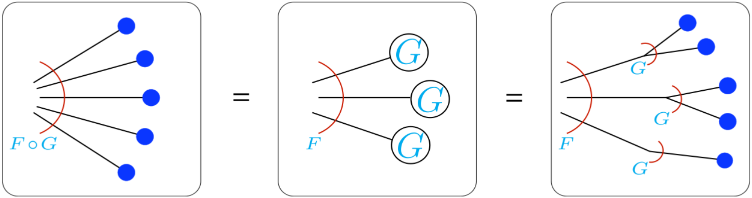


Operations on species

Composition

When $G[\emptyset] = \emptyset$,

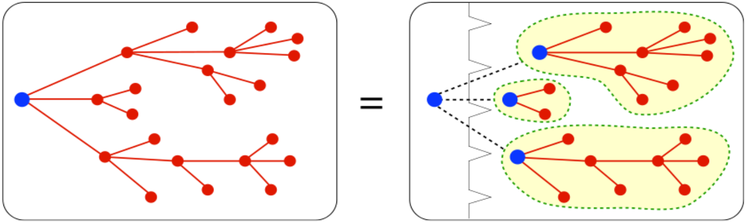
$$(F \circ G)[A] = \sum_{\pi \in P[A]} F[\pi] \times \prod_{B \in \pi} G[B]$$



Recursive definition

"A rooted tree is a root which is attached to a set of rooted trees."

$$A = X \cdot E(A)$$



Aldor-Combinat

- Started in 2006 by Ralf Hemmecke and Martin Rubey.
- Written as a fully literate program in the language of Aldor that tries to stay as close as possible to the theory of species as outlined in BLL.
- Can be found at http://www.risc.uni-linz.ac.at/people/hemmecke/aldor/combinat/

My initial work has been to port Aldor-Combinat to Sage/Python.

Demo

Future Work

Later today, I will post my code as a public patch repository.

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- Random element generation
- Lazy Karatsuba product
- Finding a recurrence
- Weighted species
- Multisort species
- Tensorial species