A First Course in Linear Algebra

An Open-Source Textbook

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Sage Developer Days 1
University of Washington
June 16, 2008

Overview

- A free (no cost) introductory textbook
- A free (GFDL'ed) introductory textbook
- Designed to encourage modification
- Designed to encourage content contributions
- A social experiment
- A disruption to traditional publishing

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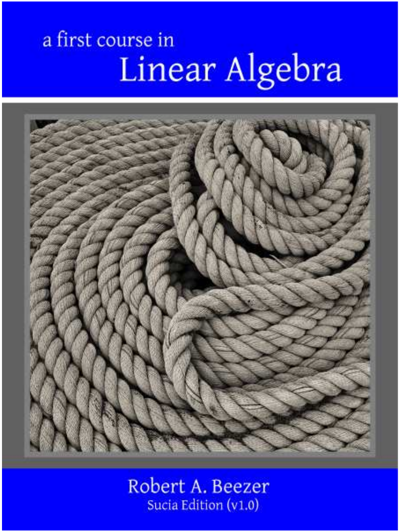
Today's Talk

- Open source
- Linear algebra
- SAGE enhanced version
- Parallels to SAGE development
- A wee bit of Python

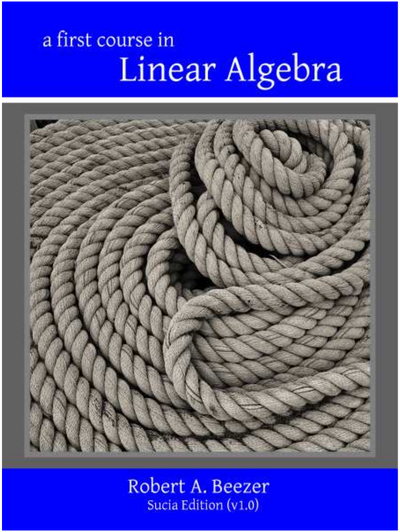
Table of Contents

- Core
  - Systems of Linear Equations
  - Vectors
  - Matrices
  - Vector Spaces
  - Determinants
  - Eigenvalues
  - Linear Transformations
  - Representations
- Topics
  - Positive Semi-Definite Matrices
  - Singular Value Decomposition
  - And more . . .
- Applications
  - Curve Fitting
  - Secret Sharing
  - ⟨Your Contribution Here⟩



By The Numbers

- 7 Chapters (Core)
- 43 Sections (Core)
- 10 Topics
- 2 Applications
- 133 Definitions
- 257 Theorems
- 382 Exercises
- 254 Solutions
- 24 Archetypes



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- Classic yellowed notes with many additions in the margins
- Moved to TEX version for convenience (and legibility)
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- Combinatorics textbook out-of-print
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- Unnecessary revisions drive up costs
- Would open-source software model carry over?
- Scratching an itch
- "The world does not need another linear algebra text, the world does needs a free linear algebra text."
- Lead by example

Short History

- September 2002: moved from paper to electronic notes
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- December 2006: Version 1.0
- January 2008: Used at Miramar College, St. Cloud State U
- Summer 2008: Version 2.0

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- Not an online course, or online textbook
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  - Internet for marketing: Google “linear algebra,” Page 1
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- Theorem and Definition flashcards
- Tarballs for LaTeX source, XML (jsMath, SAGE worksheets)

Under the Hood

- Primary Tool: LaTeX + pdflatex
- Packages: AMS, hyperref, tocloft
- Translator: tex4ht to build MathML and jsMath
- Graphics: PyX + Python to build PDF
- Sectioning: Custom macros override/extend LaTeX
- References: Custom macros override/extend LaTeX, hyperref

Acronyms

Every definition, theorem, chapter, section, figure, exercise is labeled with a mnemonic acronym (five letter maximum).

- Promotes customization (e.g. deleting sections)
- "Theorem SMZD" preferable to "Theorem 8.3.2"
- Macro: \acronymref{theorem}{SMZD} for easy referencing
- Unexpected benefits for text processing

Typical PDF

Theorem MMDAA
Matrix Multiplication Distributes Across Addition
Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then
1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$ □

Proof We'll do (1), you do (2). Entry-by-entry, for $1 \leq i \leq m$, $1 \leq j \leq p$,

$$
\begin{aligned}
[A(B+C)]_{ij} &= \sum_{k=1}^{n} [A]_{ik} [B+C]_{kj} && \text{Theorem EMP [209]} \\
&= \sum_{k=1}^{n} [A]_{ik} ([B]_{kj} + [C]_{kj}) && \text{Definition MA [192]} \\
&= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + [A]_{ik} [C]_{kj} && \text{Property DCN [713]} \\
&= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + \sum_{k=1}^{n} [A]_{ik} [C]_{kj} && \text{Property CACN [712]} \\
&= [AB]_{ij} + [AC]_{ij} && \text{Theorem EMP [209]}
\end{aligned}
$$

$$= [AB + AC]_{ij} \qquad \text{Definition MA [192]}$$

So the matrices $A(B + C)$ and $AB + AC$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [191]) we can say they are equal matrices. ■

Typical Front Matter

Section MO

VSPM	Vector Space Properties of Matrices	193
SMS	Symmetric Matrices are Square	195
TMA	Transpose and Matrix Addition	195
TMSM	Transpose and Matrix Scalar Multiplication	196
TT	Transpose of a Transpose	196
CRMA	Conjugation Respects Matrix Addition	197
CRMSM	Conjugation Respects Matrix Scalar Multiplication	197
CCM	Conjugate of the Conjugate of a Matrix	197
MCT	Matrix Conjugation and Transposes	198
AMA	Adjoint and Matrix Addition	198
AMSM	Adjoint and Matrix Scalar Multiplication	199
AA	Adjoint of an Adjoint	199

Section MM

SLEMM	Systems of Linear Equations as Matrix Multiplication	205
EMMVP	Equal Matrices and Matrix-Vector Products	207
EMP	Entries of Matrix Products	209
MMZM	Matrix Multiplication and the Zero Matrix	210
MMIM	Matrix Multiplication and Identity Matrix	211
MMDAA	Matrix Multiplication Distributes Across Addition	211
MMSMM	Matrix Multiplication and Scalar Matrix Multiplication	212
MMA	Matrix Multiplication is Associative	212
MMIP	Matrix Multiplication and Inner Products	213
MMCC	Matrix Multiplication and Complex Conjugation	213

Version 1.35

Typical XML/MathML

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&= \sum_{k=1}^{n} [A]_{ik}[B]_{kj}+[A]_{ik}[C]_{kj} && \text{Property DCN} \\
&= \sum_{k=1}^{n} [A]_{ik}[B]_{kj} + \sum_{k=1}^{n} [A]_{ik}[C]_{kj} && \text{Property CACN} \\
&= [AB]_{ij}+[AC]_{ij} && \text{Theorem EMP} \\
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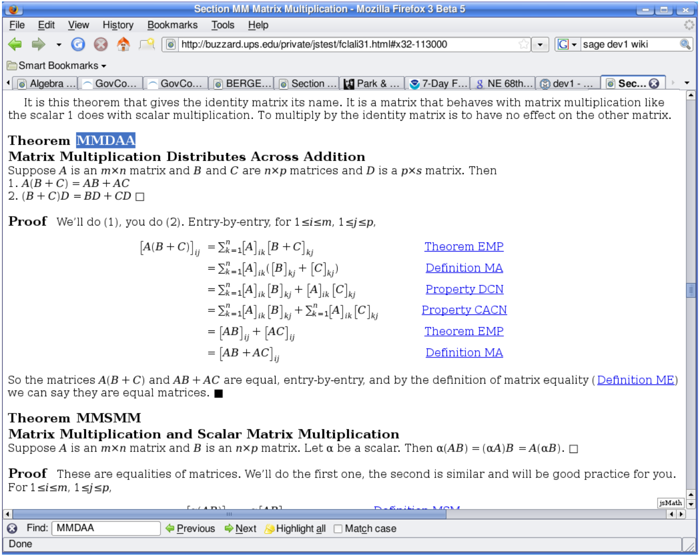
So the matrices $A(B+C)$ and $AB+AC$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices. ■

Typical jsMath



Section MM Matrix Multiplication - Mozilla Firefox 3 Beta 5

File Edit View History Bookmarks Tools Help

http://buzzard.ups.edu/private/jstest/fclali31.html#x32-113000

sage dev1 wiki

Smart Bookmarks

Algebra ... GovCo... GovCo... BERGE... Section ... Park & ... 7-Day F... NE 68th... dev1 - ... Sec...

It is this theorem that gives the identity matrix its name. It is a matrix that behaves with matrix multiplication like the scalar 1 does with scalar multiplication. To multiply by the identity matrix is to have no effect on the other matrix.

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$[A(B+C)]_{ij} = \sum_{k=1}^{n}[A]_{ik}[B+C]_{kj}$	Theorem EMP
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Theorem MMSMM
Matrix Multiplication and Scalar Matrix Multiplication
Suppose A is an $m\times n$ matrix and B is an $n\times p$ matrix. Let α be a scalar. Then $\alpha(AB)=(\alpha A)B=A(\alpha B)$. □

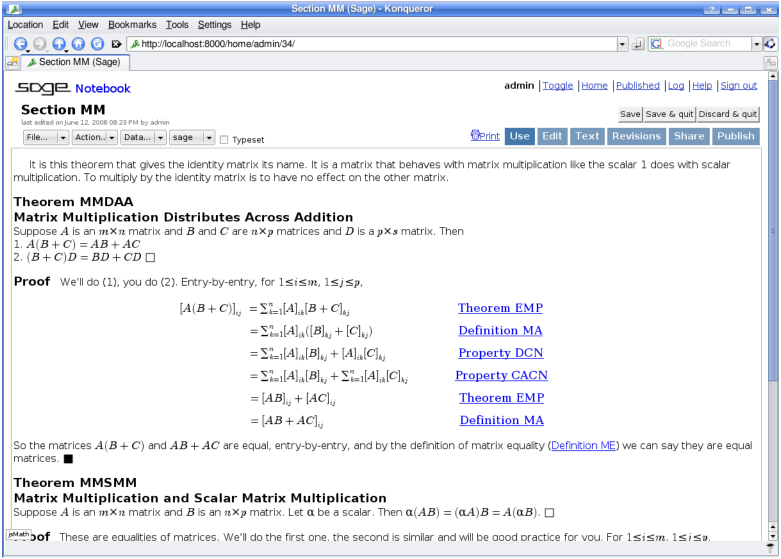
Proof These are equalities of matrices. We'll do the first one, the second is similar and will be good practice for you. For $1\le i\le m$, $1\le j\le p$,

jsMath

Find: MMDAA Previous Next Highlight all Match case

Done

Prototypical SAGE Worksheet



Section MM (Sage) - Konqueror

Location Edit View Bookmarks Tools Settings Help

http://localhost:8000/home/admin/34/

Section MM (Sage)

sage Notebook

admin | Toggle | Home | Published | Log | Help | Sign out

Section MM

last edited on June 12, 2008 08:23 PM by admin

Save | Save & quit | Discard & quit

File... | Action... | Data... | sage | Typeset

Print | Use | Edit | Text | Revisions | Share | Publish

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jsMath

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Typical Figure

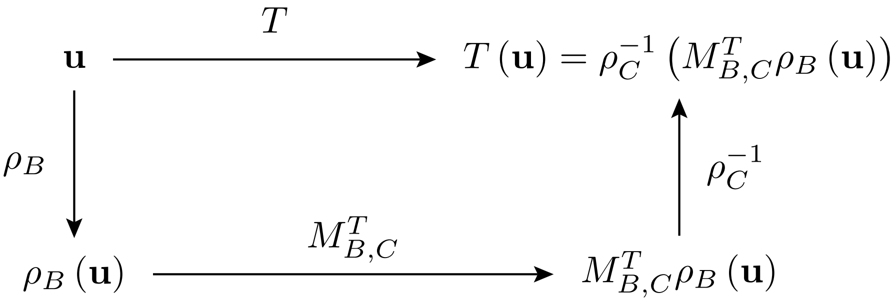

Figure Source, Python + PyX

```
def FTMRA():

  # Guides - vertical and horizontal markers
  v1=0.0
  v2=v1+0.8
  h1=0.0
  h2=h1+2.25

  c=canvas.canvas()

  # Textual areas (box names recycled from additive diagram, so not as meaningful here)
  vector = c.text(h1, v2, "$\\vect{u}$", centerboxstyle )
  represent = c.text(h1, v1, "$\\vectrep{B}{\\vect{u}}$", centerboxstyle )
  ltaction = c.text(h2, v2, "$\\lt{T}{\\vect{u}}=\\vectrepinv{C}{\\matrixrep{T}{B}{C}\\vectrep{B}{\\vect{u}}}$", centerboxstyle )
  theorem = c.text(h2, v1, "$\\matrixrep{T}{B}{C}\\vectrep{B}{\\vect{u}}$", centerboxstyle )

  # Adorned arrows
  toparrow = path.line( h1+0.5*vector.width+epsilon, v2, h2 - 0.5*ltaction.width-epsilon, v2 )
  c.stroke(toparrow, arrowheadstyle)
  c.text( 0.5*(h1+0.5*vector.width + h2 - 0.5*ltaction.width), v2+epsilon, "$T$", [text.halign.center, text.valign.baseline] )

  bottomarrow = path.line( h1+0.5*represent.width+epsilon, v1, h2 - 0.5*theorem.width-epsilon, v1 )
  c.stroke(bottomarrow, arrowheadstyle)
  c.text( 0.5*(h1+0.5*represent.width + h2 - 0.5*theorem.width), v1+epsilon, "$\\matrixrep{T}{B}{C}$", [text.halign.center, text.valign.baseline]

  leftarrow = path.line( h1, v2 - 0.5*vector.height - epsilon, h1, v1 + 0.5*represent.height+epsilon )
  c.stroke( leftarrow, arrowheadstyle )
  c.text( h1 - epsilon, 0.5*(v2 - 0.5*vector.height + v1 + 0.5*represent.height), "$\\vectrepname{B}$", [text.halign.right, text.valign.middle] )

  # Reversed the ends, and inverted label
  rightarrow = path.line( h2, v1 + 0.5*theorem.height+epsilon, h2, v2 - 0.5*ltaction.height - epsilon )
  c.stroke( rightarrow, arrowheadstyle )
  c.text( h2 + epsilon, 0.5*(v2 - 0.5*ltaction.height + v1 + 0.5*theorem.height), "$\\vectrepinvname{C}$", [text.halign.left, text.valign.middle]

  c.writePDFfile( finalpdfdir + "FTMRA.pdf" )
```

Flash Cards

Simple sed script extracts every definition and theorem

Reformat as two $4'' \times 6''$ cards per page

Theorem MMDAA Matrix Multiplication Distributes Across Addition **122**

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- Draw graph with GraphViz
- Analyze resultant graph: high degree?, root?, longest path?

Customizable

- Designed to be customized
- Extensive use of macros
  - Notation: \newcommand[1]{\transpose}{#1^ t}
  - \transpose{(AB)}=\transpose{B}\transpose{A}
- Even diagrams/figures have editable source code
- Switches control format, organization, inclusion (keyval package)
- Granular, 1250 files
- Benefits for adapting to translators
- Benefits for text processing

Extensible

- Exercises and Solutions
  - Drop-in (almost)
  - Suggested by Debian configurations

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- Still, as Benevolent Dictator, strict control on Core

Some Linear Algebra

Problem: Column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$?

Answer: All multiples of the first column, a dimension 1 subspace

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Answer: All multiples of the first column, a dimension 1 subspace

A common textbook approach:

- Column space is vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is consistent
- So row-reduce augmented matrix $[A|\mathbf{b}]$ to study solutions
- $\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & -2b_1 + b_2 \end{bmatrix}$
- $\mathbf{b}$ in column space $\iff$ system consistent $\iff$ $-2b_1 + b_2 = 0$
- Column space vectors are solutions to a homogeneous system

CAS Assistance

Row-reduce: $\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

CAS Assistance

Row-reduce: $\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Work-around: $\begin{bmatrix} 1 & 2 & 3 & b_1 & 0 \\ 2 & 4 & 6 & 0 & b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & \frac{b_2}{2} \\ 0 & 0 & 0 & 1 & -\frac{b_2}{2b_1} \end{bmatrix}$

Scale row 2: $\rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & \frac{b_2}{2} \\ 0 & 0 & 0 & -2b_1 & b_2 \end{bmatrix}$

Add last two columns:

- Second row: $-2b_1 + b_2$; equals zero $\iff$ **b** in column space
- First row: $\frac{b_2}{2} = b_1$ when **b** is in column space

CAS Assistance

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Better: $\begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 2 & 4 & 6 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$

Column space of A is null space of $\begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix}$

Extended Echelon Form

A	$m \times n$ matrix
$[A \vert I_m]$	augment with $m \times m$ identity matrix

Row-reduce to echelon form: $[A|I_m] \rightarrow [B|J]$ "extended echelon form"

Extended Echelon Form

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Row-reduce to echelon form: $[A|I_m] \rightarrow [B|J]$ "extended echelon form"

Then

- B is echelon form of A
- J is nonsingular
- J is product of elementary matrices for row operations
- $B = JA$
- $A\mathbf{x} = \mathbf{y} \iff B\mathbf{x} = J\mathbf{y}$
- A nonsingular $\Rightarrow J = A^{-1}$

Four Subspaces

$$[A|I_m] \rightarrow [B|J] = \left[\begin{array}{c|c} \text{C} & \text{K} \\ \hline 0 & \text{L} \end{array}\right]$$

Let r be the rank of A. L is an $(m-r) \times m$ matrix in echelon form and

Four Subspaces

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Let r be the rank of A. L is an $(m - r) \times m$ matrix in echelon form and

- Row space of A is row space of C
- Null space of A is null space of C

Four Subspaces

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Let r be the rank of A. L is an $(m-r) \times m$ matrix in echelon form and

- Row space of A is row space of C
- Null space of A is null space of C
- Column space of A is null space of L
- Left null space of A is row space of L

Four Subspaces

$$[A|I_m] \rightarrow [B|J] = \left[\begin{array}{c|c} \text{C} & \text{K} \\ \hline 0 & \text{L} \end{array}\right]$$

Let r be the rank of A. L is an $(m-r) \times m$ matrix in echelon form and

- Row space of A is row space of C
- Null space of A is null space of C
- Column space of A is null space of L
- Left null space of A is row space of L

Proof:

- $B = JA$ is key tool
- No dimension arguments
- No vector space properties of subspaces
- Corollary: Subspace dimensions in terms of n, m and r

Open-Source Benefits

- No deadlines!
- Rapid correction of typos and frequent releases
- Unparalled navigation in electronic editions (no DRM!)
- Text processing of source for derivative supplements
- Classroom-inspired exercises
- Student contributions
- tex4ht: LaTeX→XML→Braille
- Introduced to SAGE (contributed Computation Notes)
- Interesting contributors
  - Frenchman correcting my discourse on “liberte” versus “gratis”
  - Retired South African mining engineer coding `echelon_form()` routine
  - Indonesian lecturer commenting on finer points of a proof

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  - SAGE code: LaTeX source→jsMath→Worksheet
- Automate: creating "textbook" exercises
  - SAGE code
  - Web form interface
  - Request rows, columns, rank, specific pivot columns
  - Output matrices with "nice" bases for all 4 subspaces
  - Or, request Jordan canonical form