# Computing Hecke Operators On Drinfeld Cusp Forms

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# Notation

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- $\mathbb{F}_q$ with $q = p^r$ finite field with $q$ elements
- $A := \mathbb{F}_q[T]$
- $K := \mathrm{Quot}(A)$
- $v_\infty$ valuation of K at the place $\infty$, i.e.
  $v_\infty(\frac{f}{g}) = \deg(g) - \deg(f), v_\infty(0) = \infty$
- $K_\infty$ the completition of $K$ at $v_\infty$, i.e. $K_\infty = \mathbb{F}_q((\pi_\infty))$ the Laurent series field where $\pi_\infty$ is the uniformizer $T^{-1}$.
- $\mathcal{O}_\infty := \{x \in K_\infty \mid v_\infty(x) \geq 0\}$

# The Bruhat-Tits tree $\mathcal{T}$

## Definition of $\mathcal{T}$

- Let $\mathrm{Vert}(\mathcal{T})$ be the equivalence classes of $\mathcal{O}_\infty$-lattices in $K_\infty^2$. Each such equivalence class defines a vertex of $\mathcal{T}$.
- Let $\Lambda, \Lambda' \in \mathrm{Vert}(\mathcal{T})$ and choose a lattice $L \in \Lambda$. $\Lambda$ and $\Lambda'$ are connected in $\mathcal{T}$ iff there exists a $L' \in \Lambda'$ such that $L' \subseteq L$ and $L/L' \simeq \mathcal{O}_\infty/\pi_\infty\mathcal{O}_\infty$. The set of directed edges of $\mathcal{T}$ is called $\mathrm{Edge}(\mathcal{T})$.
- Let $|\mathcal{T}|$ be the geometric realization of $\mathcal{T}$.

## Theorem about the structure of $\mathcal{T}$

$\mathcal{T}$ is a $q+1$-regular tree, i.e. $\mathcal{T}$ is a connected, cycle-free tree, where every vertex has $q+1$ neighbours.

# Example for $q = 3$

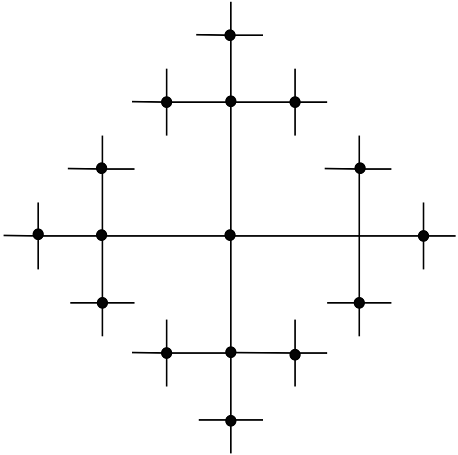

Figure: The Bruhat-Tits tree for $\mathbb{F}_3$

# Operation of $\mathrm{GL}_2(A)$ on $\mathcal{T}$

- There is a bijection

$$\mathrm{Vert}(\mathcal{T}) \longrightarrow \mathrm{GL}_2(K_\infty)/\,\mathrm{GL}_2(\mathcal{O}_\infty)K_\infty^\star$$

- There is a bijection

$$\mathrm{Edge}(\mathcal{T}) \longrightarrow \mathrm{GL}_2(K_\infty)/\Gamma_\infty K_\infty^\star$$

with $\Gamma_\infty := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) \mid v_\infty(c) > 0\}$

- $\mathrm{GL}_2(A)\backslash\mathcal{T}$ is just a half-line.
  Reason: $\mathrm{GL}_2(A)\backslash\,\mathrm{GL}_2(K_\infty)/\,\mathrm{GL}_2(\mathcal{O}_\infty)K_\infty^\star \cong \{\begin{pmatrix} 1 & 0 \\ 0 & \pi^n \end{pmatrix} \mid n \in \mathbb{N}\}$
- We write $\Lambda_n$ for the class of the lattice $\mathcal{O}_\infty \oplus \pi_\infty^n \mathcal{O}_\infty$.

# Congruence subgroups

Let $N \in A$ be normalized.

- $\Gamma(N) := \{\gamma \in \mathrm{GL}_2(A) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}$
- A subgroup of $\mathrm{GL}_2(A)$ containig $\Gamma(N)$ for any $N \in A$ is called a congruence subgroup.
- $\Gamma_0(N) := \{\gamma \in \mathrm{GL}_2(A) \mid \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod N\}$
- $\Gamma_1(N) := \{\gamma \in \mathrm{GL}_2(A) \mid \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N\}$
- Congruence subgroups are of finite index in $\mathrm{GL}_2(A)$, since $\Gamma(N)\backslash \mathrm{GL}_2(A) \cong \begin{pmatrix} \mathbb{F}_q^\star & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(A/N)$.

# Calculation of $\Gamma\backslash\mathcal{T}$ with $\Gamma$ a congruence subgroup, Idea

- $\mathrm{GL}_2(A)\backslash\mathcal{T}$ is a simple half line.
- $\mathrm{GL}_2(A)\backslash\mathcal{T} : \Lambda_0 \rightarrow \Lambda_1 \rightarrow \Lambda_2 \rightarrow \dots$
- $\Gamma\backslash\mathcal{T}$ is a covering of $\mathrm{GL}_2(A)\backslash\mathcal{T}$
- Elements of $\Gamma\backslash\mathcal{T}$ are $\Gamma$-orbits of $\mathcal{T}$. Every $\mathrm{GL}_2(A)$-orbit of $\mathcal{T}$ decomposes into finitly many $\Gamma$-orbits, since $\Gamma\backslash\mathrm{GL}_2(A)$ is finite.
- We need to know $\mathrm{Stab}_{\mathrm{GL}_2(A)}(\Lambda_i)$ to see how an $\mathrm{GL}_2(A)$-orbit decomposes.

# Algorithm for the calculation of $\Gamma\backslash\mathcal{T}$

- Let $G_i := \mathrm{Stab}_{\mathrm{GL}_2(A)}(\Lambda_i)$. A simple calculation shows, that $G_0 = \mathrm{GL}_2(\mathbb{F}_q)$ and $G_i = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{F}_q^\star, b \in \mathbb{F}_q[T], \deg(b) \leq i\}$.
- Let $S = \{s_1, \ldots, s_n\}$ be a set of representativs of $\Gamma\backslash\mathrm{GL}_2(A)$.
- Let $\Upsilon$ be the standard half line $\Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \ldots$ and $s_i(\Upsilon)$ the halfline $s_i(\Lambda_0) \to s_i(\Lambda_1) \to s_i(\Lambda_2) \to \ldots$
- Then $\Gamma\backslash\mathcal{T}$ can be obtained by taking the halflines $s_1(\Upsilon), \ldots, s_m(\Upsilon)$ and identify vertices and edges using the following rules:
  1. Only identify vertices and edges of the same level.
  2. $s_i(\Lambda_n) \sim s_j(\Lambda_n)$ iff there exists a $g \in G_n$ such that $s_i g s_j^{-1} \in \Gamma$.
  3. $s_i((\Lambda_0, \Lambda_1)) \sim s_j((\Lambda_0, \Lambda_1))$ iff there exists a $g \in G_0 \cap G_1$ such that $s_i g s_j^{-1} \in \Gamma$.
  4. $s_i((\Lambda_n, \Lambda_{n+1})) \sim s_j((\Lambda_n, \Lambda_{n+1}))$ iff there exists a $g \in G_n$ such that $s_i g s_j^{-1} \in \Gamma$ for $n \geq 1$.
  5. Once $n \geq \deg(N)$ we stop; the remaining half-lines represent cusps.

# Algorithm for the calculation of $\Gamma\backslash\mathcal{T}$ II

1. Compute a canonical representative system $S = \Gamma\backslash \mathrm{GL}_2(A)$.
2. First compute the edges over $(\Lambda_0, \Lambda_1)$: Let $B(\mathbb{F}_q) := G_0 \cap G_1$. For each $s_i \in S$ loop over $\gamma \in B(\mathbb{F}_q)$ and compute the canonical representative $s_j$ of $s_i\gamma$. Remove $s_j$ from $S$ and add $(s_j, \gamma)$ to an label of the edge $s_i(\Lambda_0, \Lambda_1)$.
3. Next compute the vertices over $\Lambda_0$: Let $S'$ be the remaining representatives from step 2, so $S' = \Gamma\backslash \mathrm{GL}_2(A)/B(\mathbb{F}_q)$. Now for each $s_i \in S'$ loop over all $\gamma' \in \mathbb{P}^1(\mathbb{F}_q) = \mathrm{GL}_2(\mathbb{F}_q)/B(\mathbb{F}_q)$, compute the representative $s_j$ of $s_i\gamma'$. Remove $s_j$ from $S'$ and add $(s_j, \gamma\gamma')$ to an label of the vertex $s_i\Lambda_0$.
4. Next compute the vertices over $\Lambda_i$, $i = 1, \ldots, \deg(N) - 1$: For the fiber over $\Lambda_1$ start again with $S' = \Gamma\backslash \mathrm{GL}_2(A)/B(\mathbb{F}_q)$ and loop over $G_1/B(\mathbb{F}_q)$. For the fiber over $\Lambda_i, i \geq 2$ start with the remaining representatives from step $i-1$ and loop over $G_i/G_{i-1}$. Change the vertex labels as in step 3.

# Example: $q = 2$, $\Gamma_1(T^2)\backslash\mathcal{T}$

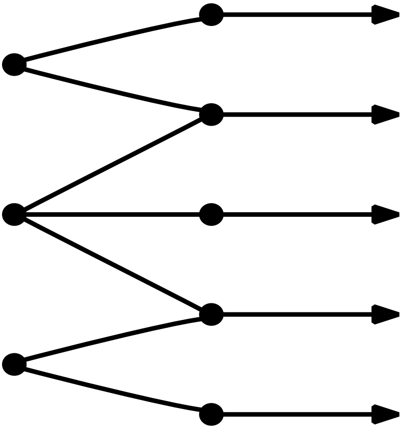

Figure: The quotient $\Gamma_1(T^2)\backslash\mathcal{T}$ for $\mathbb{F}_2$

# Example: $q = 3$, $\Gamma_1(T^2)\backslash\mathcal{T}$

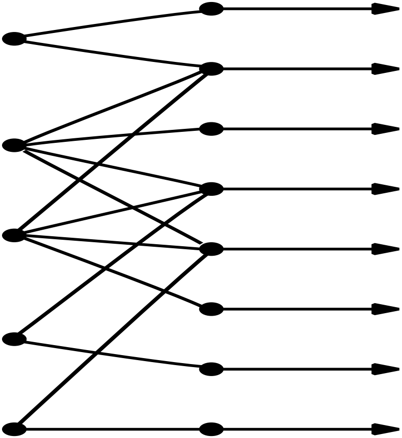

Figure: The quotient $\Gamma_1(T^2)\backslash\mathcal{T}$ for $\mathbb{F}_3$

# The Drinfeld upper half plane

## The Drinfeld upper half plane

- Let $\mathbb{C}_\infty := \widehat{K_\infty^{\mathrm{alg}}}$.
- $\Omega := \mathbb{P}^1(\mathbb{C}_\infty) \backslash \mathbb{P}^1(\mathbb{K}_\infty)$ is called the Drinfeld upper half plane.
- $\mathrm{GL}_2(K_\infty)$ acts on $\Omega$ by fractional linear transformations.

# Drinfeld modular forms

There are natural inclusions

$$\mathrm{GL}_2(A) \hookrightarrow \mathrm{GL}_2(K) \hookrightarrow \mathrm{GL}_2(K_\infty).$$

Let $\Gamma \subset \mathrm{GL}_2(A)$ be a congruence subgroup.

## Definition: Drinfeld modular form (Goss)

A Drinfeld modular form of weight $k$ (and trivial type) for $\Gamma$ is a rigid analytic function

$$f : \Omega \to \mathbb{C}_\infty$$

such that

1. $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
2. $f$ has a Laurent series expansion at all cusps of $\Gamma$ with vanishing principal part.

# Definition of harmonic cocycles

- For an edge $e \in \mathrm{Edge}(\mathcal{T})$ let $e^\star$ denote the same edge with orientation reversed.
- For an vertex $v \in \mathrm{Vert}(\mathcal{T})$ we write $e \mapsto v$ if $v$ is the target of $e$
- Let $M$ be any vector space with a $\mathrm{GL}_2(A)$-operation and $\Gamma \subseteq \mathrm{GL}_2(A)$ a congruence subgroup.
  1. A function $c : \mathrm{Edge}(\mathcal{T}) \longrightarrow M$ is called an $M$-valued harmonic cocycle, if
     1. for all vertices $v \in \mathrm{Vert}(\mathcal{T})$ we have $\sum_{e \mapsto v} c(e) = 0$.
     2. $c(e^\star) = -c(e)$ for all edges $e \in \mathrm{Edge}(\mathcal{T})$.
  2. A function $c : \mathrm{Edge}(\mathcal{T}) \longrightarrow M$ is called $\Gamma$-equivariant, if for all $\gamma \in \Gamma$ we have $c(\gamma e) = \gamma c(e)$.
  3. Let $C_{\mathrm{har}}(\Gamma, M)$ be the vector space of $M$-valued $\Gamma$-equivariant harmonic cocycles with values in $M$.

# Automatic cuspidality

A $\Gamma$-equivariant harmonic cocycle $c$ is called cuspidal if there exist a finite subgraph $Z \subset \Gamma\backslash\mathcal{T}$ with $c(e) = 0$ for all $e \notin Y(\pi^{-1}(Z))$.

**Theorem: Automatic cuspidality (Teitelbaum)**

Let $M$ be a finite-dimensional vector-space over a field of characteristic $p$ with a $\mathrm{GL}_2(A)$-operation. Then every $M$-valued $\Gamma$-equivariant harmonic cocycle is cuspidal.

# Connection with Drinfeld cusp forms

Recall:

- Drinfeld modular forms are rigid analytic functions on $\Omega$.
- $\Omega$ is a tubular neigborhood of $|\mathcal{T}|$ via $\rho$.
- $\rho^{-1}$ of the inner part of an edge $e$ is an annulus $A(e)$.

For $f$ of weight 2 define the map

$$\mathrm{Res}_2(f) : \mathrm{Edge}(\mathcal{T}) \to \mathbb{C}_\infty : e \mapsto \mathrm{Res}_{A(e)}(f dz).$$

**Theorem (Teitelbaum)**

$\mathrm{Res}_2$ defines an isomorphism from the $\mathbb{C}_\infty$-vector space of Drinfeld cusp forms of weight 2 and level $\Gamma$ to $C_{\mathrm{har}}(\Gamma, K) \otimes_K \mathbb{C}_\infty$. An analog result holds for weight $k$ with $M = (\mathrm{Sym}^{k-2}((K^2)^\star))^\star \otimes_K \mathbb{C}_\infty$.

# Stable Edges

- From now on let $\Gamma$ be one of $\Gamma_1(N)$, $\Gamma(N)$ or $\Gamma_0(N) \cap \Gamma_1(P)$, i.e. $\Gamma$ is $p'$-torsion-free.
- An edge $e \in \mathrm{Edge}(\mathcal{T})$ (or a vertex $v \in \mathrm{Vert}(\mathcal{T})$) is called $\Gamma$-stable, if $\mathrm{Stab}_\Gamma(e) = \{1\}$ (or $\mathrm{Stab}_\Gamma(v) = \{1\}$). (So, i.e. there are no $\mathrm{GL}_2(A)$-stable edges!)
- Fact: The stable part of the tree is connected in $\Gamma \backslash \mathcal{T}$.
- Fact: A vertex $v \in \mathrm{Vert}(\mathcal{T})$ is stable if and only if its image in $\Gamma \backslash \mathcal{T}$ has exactly $q+1$ neighbours. An edge $v \in \mathrm{Edge}(\mathcal{T})$ is stable, if and only if one of the adjacent vertices is stable (except for the case $\Gamma_1(T)$).
- Fact: For every unstable edge $e \in \mathrm{Edge}(\mathcal{T})$ there is a finite and easy to compute set $\mathrm{Source}(e)$ of stable edges of $\mathcal{T}$ such that

$$c(e) = \sum_{e' \in \mathrm{Source}(e)} c(e').$$

# Stable Edges, cont.

- So a harmonic cocycle $c$ is determined by the values of $c$ on the stable part of $\Gamma\backslash\mathcal{T}$.
- Let $n = \deg(N)$. Then an edge in the covering over $(\Lambda_i, \Lambda_{i+1})$ with $i \geq n$ is unstable.
- In fact a harmonic cocycle is determined by the values of $c$ on the stable edges over the edge $(\Lambda_0, \Lambda_1)$, and for every stable vertex over $\Lambda_0$ we get one relation between these edges.

# Example: $q = 2$, $\Gamma_1(T^2)\backslash\mathcal{T}$

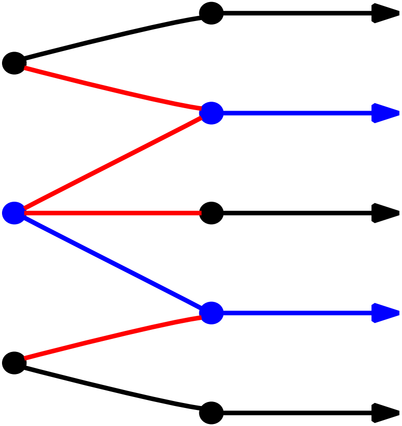

Figure: Colored: Stable edges and vertices. Red: Minimal set of edges, that determine a harmonic cocycle.

# Example: $q = 3$, $\Gamma_0(T^2(T+1)) \cap \Gamma_1(T) \backslash \mathcal{T}$

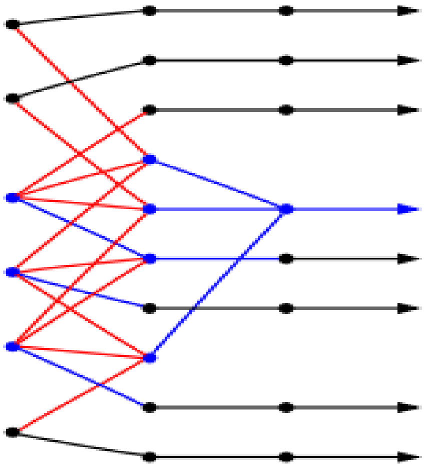

Figure: Colored: Stable edges and vertices. Red: Minimal set of edges, that determine a harmonic cocycle.

# Construction of an explicit basis of $C_{\text{har}}(\Gamma, M)$

- Let $\mathcal{B} := \{w_1, \ldots, w_m\}$ be a basis of $M$.
- Let $S := \{e_1, \ldots, e_r\} \subset \text{Edge}(\Gamma\backslash\mathcal{T})$ be the set of stable edges over $(\Lambda_0, \Lambda_1)$ and let $\{v_1, \ldots, v_s\} \subset \text{Vert}(\Gamma\backslash\mathcal{T})$ be the set of stable vertices over $\Lambda_0$.
- For each $v_i$ choose one edge $e_{v_i}$ in $S$ having $v_i$ as its origin. Let $BS$ be the set of edges obtained by removing all $e_{v_i}$ from $S$. We call these edges stable basis edges.
- For each $e \in BS$ choose one fixed lift $\tilde{e} \in \text{Edge}(\mathcal{T})$.
- For each tupel $(e, w) \in BS \times \mathcal{B}$ let $c_{e,w}$ be the harmonic cocycle defined by $c_{e,w}(\tilde{e}) := w$ and $c_{e,w}(\tilde{e}') = 0$ for all other $e' \in BS$.
- Note that $c_{e,v}$ extends uniquely to all $e \in \text{Edge}(\mathcal{T})$.

**Proposition**

The cocycles $\{c_{e,v} \mid e \in BS, v \in \mathcal{B}\}$ form a basis of $C_{\text{har}}(\Gamma, M)$.

# Hecke operators on $C_{har}(\Gamma, M)$

- Let $\mathfrak{p} \in A$ be a prime ideal. Translating the Hecke action to the tree gives:

$$T_{\mathfrak{p}}(c)(e) = \sum_{\delta \in (\Gamma \cap \Gamma_0(\mathfrak{p}))\backslash\Gamma} \delta^{-1} \begin{pmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}^{-1} c(\begin{pmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \delta e)$$

- Let $e = (\gamma\Lambda_0, \gamma\Lambda_1)$ be given. To evaluate $c(\begin{pmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \delta e)$ we consider the matrices $\begin{pmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \delta\gamma$ and $\begin{pmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \delta\gamma \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$.
- Writing these two matrices in the form $\gamma' \begin{pmatrix} 1 & 0 \\ 0 & \pi^{k_i} \end{pmatrix} \alpha \quad (i \in \{1,2\})$ with $\alpha \in K_\infty^\star \mathrm{GL}_2(\mathcal{O}_\infty)$ and $\gamma' \in \mathrm{GL}_2(A)$ we find the image edge $\begin{pmatrix} \mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \delta e = \gamma'(\Lambda_{k_1}, \Lambda_{k_2})$. Note that $|k_1 - k_2| = 1$.
- Write $\gamma' = \gamma_0 s_j$ with $\gamma_0 \in \Gamma$ and $s_j \in S$ and use the $\Gamma$-equivariance of $c$ to obtain an edge in the pre-stored quotient graph $\Gamma\backslash\mathcal{T}$.

# Computing the matrix of $T_\mathfrak{p}$ w.r.t. our basis

- To compute a matrix for $T_\mathfrak{p}$ we need to loop over all $c_{e,v}$ with $e \in BS$ and $v \in \mathcal{B}$.
- Let $c_{e,v}$ be given. Let $e' \in BS$ be another stable basis edge. We need to compute the value $T_\mathfrak{p}(c_{e,v})(e')$.
- As described in the last slide we are reduced to summing over values of the form $\gamma c(e'')$ where $e''$ is some other edge in the quotient graph.
- Note that $c_{e,v}$ is zero everywhere outside of the preimage of two half-lines in $\Gamma\backslash\mathcal{T}$.
- If $e''$ is contained in the preimage of one of these half-lines, then if $e''$ is stable, we know the value $c_{e,v}(e'')$. If $e''$ is unstable, we have to sum over the source of $e''$. All the edges in the source are still in the preimage of the same half-line.

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