Sage Days 18, Clay Mathematics Institute

Fundamental domains for Shimura curves and computation of Stark-Heegner points

Matthew Greenberg, University of Calgary

04.12.2009

Outline

1. Introduction: What are Stark-Heegner points?

2. Darmon's construction and conjecture

3. a simplified construction in the "genus zero tame level" case

   - p-adic and cohomological machinery

   - computing Stark-Heegner points

Introduction

- Stark-Heegner points are points on modular elliptic curves.
- They are constructed p-adic analytically.
- They are conjectured to be rational over specific class fields of *real* quadratic fields.
- Their rationality falls outside the scope of complex multiplication (CM) theory which is concerned with class fields of *imaginary* quadratic fields.
- Stark-Heegner points are to Mordell-Weil groups as Stark units are to unit groups.

Darmon's construction

- H. Darmon, *Integration of $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications*, Annals of Math., 2001.

- input:

  1. an elliptic curve $E/\mathbb{Q}$ of conductor pN with $p \nmid N$

  2. a *real* quadratic field K of with $(\operatorname{disc} K, pN) = 1$ satisfying the *Stark-Heegner hypothesis* — p is inert in K & all $\ell \mid N$ split in K

- output: a system of local points $P_\psi \in E(K_p)$ indexed by optimal embeddings $\psi : \mathcal{O} \longrightarrow R_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2(\mathbb{Z}) : N|c \right\}$, where $\mathcal{O}$ is in order in K of conductor prime to pN

A generalization

- input:

  1. semistable $E/\mathbb{Q}$ of conductor pN with $p \nmid N$

  2. a real quadratic field K of with $(\operatorname{disc} K, pN) = 1$ such that

$$\left(\frac{\operatorname{disc} K}{p}\right) = -1 \text{ and } \operatorname{sign}(L(E/K, s)) = -1.$$

Set

$$N^{\pm} := \prod \left\{ \ell : \ell | N, \left(\frac{\operatorname{disc} K}{\ell}\right) = \pm 1 \right\},$$

$R_0^{N^-}(N^+) :=$ Eichler order of level N^+ in indefinite quaternion algebra of discriminant N^-.

- output: a system of local points $P_\psi \in E(K_p)$ indexed by optimal embeddings $\psi : \mathcal{O} \longrightarrow R_0^{N^-}(N^+)$, where $\mathcal{O}$ is in order in K of conductor prime to pN

We conjecture that the *Stark-Heegner points* P_ψ behave like classical Heegner points:

- P_ψ is rational over the ring class field $H_\mathcal{O}$ associated to the order $\mathcal{O}$.

- They obey a Shimura-style reciprocity law which gives an analytic description of the Galois action on the P_ψ.

Computing Stark-Heegner points

- Darmon-Green and Darmon-Pollack collected much numerical evidence for this conjecture in the case $N^+ = N^- = 1$.

- I would like to describe an (unimplemented!) approach for doing analogous computations in the case where the Shimura curve $X_0^{N^-}(N^+)$ has genus zero.

The Stark-Heegner construction

Let

$$\Gamma = R_0^{N^-}(N^+)_1^\times, \quad \Gamma_0 = R_0^{N^-}(pN^+)_1^\times$$

$$\Theta = \Gamma *_{\Gamma_0} \Gamma.$$

- Θ is a p-arithmetic group. It acts on $\mathcal{H}$ with dense orbits, but discontinuously on $\mathcal{H}_p \times \mathcal{H}$.
- One can formulate a theory of modular forms on Θ, a p-adic theory of integration and periods of such forms.
- We have a formalism in which Stark-Heegner points arise by integrating modular forms on Θ over non-closed cycles, yielding invariants well defined modular the (Tate) period lattice of $E/\mathbb{Q}_p$.

A simplified construction in a special case

- We have good algorithms for computing with Fuchsian groups, Θ is not a Fuchsian group and I don't know how to compute with modular forms on Θ.

- If Γ^{ab} is finite, one can construct the formal group logarithms of the Stark-Heegner points without using Θ.

- This is motivated my (still unrealized) desire to compute Stark-Heegner points arising from Shimura curve parametrizations.

I would like to describe this simplified construction by analogy:

- recall major players in the classical Heegner point construction

- identify p-adic analogues in the Stark-Heegner world

Heegner points

setting: $E/\mathbb{Q}$ of conductor N, $f \in S_2(\Gamma)$ with $L(E,s) = L(f,s)$

Let $p : \mathcal{H}^* \to X_0(N)(\mathbb{C})$ be the natural projection. The integration map

$$H^0(X_0(N)(\mathbb{C}), \Omega^1_{\mathsf{hol}}) \times \operatorname{Div}^0 \mathcal{H}^* \longrightarrow \mathbb{C}, \quad (\omega, D) \mapsto \int_D p^*\omega$$

induces a map

$$\int_{-} 2\pi i f(\tau) d\tau : H_0(\Gamma, \operatorname{Div}^0 \mathcal{H}^*) \longrightarrow \mathbb{C}/\Lambda_f \cong E(\mathbb{C}).$$

Theorem. If $D \in \operatorname{Div}^0 \mathcal{H}^*$ is supported on *imaginary quadratic irrationalities* and cusps, then the image in $E(\mathbb{C})$ of $\int_D 2\pi i f(\tau) d\tau$ lies in $E(\bar{\mathbb{Q}})$.

- input:

  1. an elliptic curve $E/\mathbb{Q}$

  2. a divisor D supported on imaginary quadratic irrationalities and cusps

- output: a complex number $\int_D 2\pi i f(\tau) d\tau$ whose image under the Weierstrass uniformization of E belongs to $E(\bar{\mathbb{Q}})$

Can we naively replace D by a divisor supported on real quadratic irrationalities and expect to get algebraic points?

No! There are no real quadratic irrationalities in $\mathcal{H}^*$.

Enter p-adic analysis

- There are plenty of imaginary quadratic irrationalities in the *p-adic upper half-plane*

$$\mathcal{H}_p = \mathbb{P}^1(\mathbb{Q}_{p^2}) - \mathbb{P}^1(\mathbb{Q}_p).$$

  If K is a real quadratic field and p is inert in K, then $K \cap \mathcal{H}_p \neq \emptyset$.

- Replace all archimedean objects with their nonarchimedean counterparts:

$$\mathcal{H} \longleftrightarrow \mathcal{H}_p, \quad \Omega^1_{\mathrm{hol}}(\mathcal{H}) \longleftrightarrow \Omega^1_{\mathrm{rig}}(\mathcal{H}_p), \quad 2\pi i f(\tau)d\tau, \longleftrightarrow ???$$

  Here, $\Omega^1_{\mathrm{rig}}(\mathcal{H}_p)$ is the group of rigid-analytic differential 1-forms on $\mathcal{H}_p$.

Eichler-Shimura isomorphisms

- naive idea: Replace

$$2\pi i f(\tau) d\tau \in H^0(\Gamma_0, \Omega^1(\mathcal{H})),$$

  with an element of $H^0(\Gamma_0, \Omega^1_{\text{rig}}(\mathcal{H}_p))$.

- There is no relation between $S_2(\Gamma_0)$ and $H^0(\Gamma_0, \Omega^1_{\text{rig}}(\mathcal{H}_p))$ (that I know of).

- But there *is* a relation between $S_2(\Gamma_0)$ and $H^1(\Gamma, \Omega^1_{\text{rig}}(\mathcal{H}_p))$.

- To related them we use an intermediate object — $H^1(\Gamma_0, \mathbb{C}_p) = H^1(\Gamma_0, \mathbb{Q}) \otimes \mathbb{C}_p$.

Theorem. (G. Stevens) Let $f \in S_2(\Gamma_0)^{p\text{-new}}$ be a Hecke eigenform with rational Hecke eigenvalues. Then

$$H^1(\Gamma, \Omega^1_{\mathrm{rig}}(\mathcal{H}_p))^{f,\pm} \xrightarrow{\mathrm{res}_*} H^1(\Gamma_0, \mathbb{C}_p)^{f,\pm}$$

is an isomorphism.

$$\begin{array}{ccc} H^1(\Gamma_0, \mathbb{Q})^{f,\pm} & \hookrightarrow & H^1(\Gamma, \Omega^1_{\mathrm{rig}}(\mathcal{H}_p))^{f,\pm} \\ \varphi^{\pm} & \mapsto & \Phi^{\pm} \end{array}$$

(Define $\Phi^{\pm}$ by this diagram.)

- $\Phi^{\pm}$ plays the role of $2\pi i f(\tau) d\tau$ in the Stark-Heegner point construction.

- I want to sketch the proof of the above theorem.

Locally analytic functions

- Let $\mathcal{C}_{\text{la}}(\mathbb{P}^1(\mathbb{Q}_p))$ be the group of locally analytic functions on $\mathbb{P}^1(\mathbb{Q}_p)$ with values in $\mathbb{C}_p$:

  - Let $\mathcal{B}_n$ be decomposition of $\mathbb{P}^1(\mathbb{Q}_p)$ into $p^n + p^{n-1}$ residue disks of radius p^{-n}, i.e., the fibres of $\mathbb{P}^1(\mathbb{Q}_p) \to \mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z})$.

  - Let $\mathcal{C}_{\text{la}}(\mathbb{P}^1(\mathbb{Q}_p), n)$ be the group of functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which have convergent power series expansions on each disk in $\mathcal{B}_n$.

  - $\mathcal{C}_{\text{la}}(\mathbb{P}^1(\mathbb{Q}_p)) = \varinjlim \mathcal{C}_{\text{la}}(\mathbb{P}^1(\mathbb{Q}_p), n)$

- $\mathbb{C}_p$ embeds in $\mathcal{C}_{\text{la}}(\mathbb{P}^1(\mathbb{Q}_p))$ as constant functions.

Locally analytic distributions

- $\mathcal{D}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p), n) = \mathcal{C}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p), n)^\vee$,

- $\mathcal{D}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p)) = \mathcal{C}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p))^\vee = \varprojlim \mathcal{D}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p), n)$

- $\mathcal{D}^0_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p)) = (\mathcal{C}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{C}_p)^\vee$

Theorem. (Morita duality) There is a perfect $\mathsf{GL}_2(\mathbb{Q}_p)$-equivariant pairing

$$\mathcal{C}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{C}_p \times \Omega^1_{\mathsf{rig}}(\mathcal{H}_p) \longrightarrow \mathbb{C}_p.$$

- Thus, $\Omega^1_{\mathsf{rig}}(\mathcal{H}_p) \cong \mathcal{D}^0_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p))$.

Proof that

$$H^1(\Gamma, \Omega^1_{\mathrm{rig}}(\mathcal{H}_p))^f \xrightarrow{\mathrm{res}_*} H^1(\Gamma_0, \mathbb{C}_p)^f$$

is an isomorphism:

- We use the isomorphism $\Omega^1_{\mathrm{rig}}(\mathcal{H}_p) \cong \mathcal{D}^0_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p))$.

- The following diagram commutes:

$$\begin{array}{ccc} \Omega^1_{\mathrm{rig}}(\mathcal{H}_p) & \xrightarrow{\quad \mathrm{res} \quad} & \mathbb{C}_p \\ \downarrow & \nearrow_{\int_{X_\infty}} & \\ \mathcal{D}^0_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p)) & & \end{array}$$

  Here, $X_\infty \in \mathcal{B}_1$ is the residue disk around infinity of radius p^{-1}.

- Shapiro's lemma:

$$H^1(\Gamma, \mathcal{D}_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p))) \cong H^1(\Gamma_0, \mathcal{D}_{\mathrm{la}}(X_\infty))).$$

- Stevens shows that

$$H^1(\Gamma_0, \mathcal{D}_{\mathrm{la}}(X_\infty))^f \xrightarrow{\mathrm{res}_*} H^1(\Gamma_0, \mathbb{C}_p)^f$$

  is an isomorphism for all eigenforms $f \in S_2(\Gamma_0)$ of slope < 1.

- SES: $0 \to \mathcal{D}^0_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p)) \longrightarrow \mathcal{D}_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p)) \longrightarrow \mathbb{C}_p \to 0$

- LES:

$$0 = H^2(\Gamma, \mathbb{Q}) \longrightarrow H^1(\Gamma, \mathcal{D}^0_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p))) \longrightarrow H^1(\Gamma, \mathcal{D}_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p))) \longrightarrow H^1(\Gamma, \mathbb{C}_p) = 0$$

Computing $\Phi^{\pm}$

- $\varphi^{\pm}$ can be computed using fundamental domain algorithms or modular symbols.

- Irritating to compute $\Phi^{\pm} \in H^1(\Gamma, \Omega^1_{\mathrm{rig}}(\mathcal{H}_p))$ (The problem is the coefficient module $\Omega^1_{\mathrm{rig}}(\mathcal{H}_p) \cong \mathcal{D}^0_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p))$.)

- But, it can be shown that the construction can be accomplished using the image of $\Phi^{\pm}$ under

$$\begin{aligned} H^1(\Gamma, \Omega^1_{\mathrm{rig}}(\mathcal{H}_p)) &\cong H^1(\Gamma, \mathcal{D}^0_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p))) \\ &\longrightarrow H^1(\Gamma, \mathcal{D}^0_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p), 1)). \end{aligned}$$

- We *can* efficiently represent elements of $\mathcal{D}_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p), 1)$ using moments and the *Pollack-Stevens filtrations*.

Where were we?

- Replace all archimedean objects with their nonarchimedean counterparts:

$$\mathcal{H} \longleftrightarrow \mathcal{H}_p, \quad \Omega^1_{\mathsf{hol}}(\mathcal{H}) \longleftrightarrow \Omega^1_{\mathsf{rig}}(\mathcal{H}_p),$$

$$2\pi i f(\tau) d\tau \in H^0(\Gamma, \Omega^1_{\mathsf{hol}}(\mathcal{H})) \longleftrightarrow \Phi^{\pm} \in H^1(\Gamma, \Omega^1_{\mathsf{rig}}(\mathcal{H}_p))^f$$

$D \in \mathrm{Div}^0 \mathcal{H}$ supported on imaginary quadratic irrationalities $\longleftrightarrow$???

- Whatever ??? is in our p-adic context, we should be able to pair it with $\Phi^{\pm}$ to get a point.

- There is a natural "evaluation" pairing

$$H^1(\Gamma, \mathcal{D}^0_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p))) \times H_1(\Gamma, \mathcal{C}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{C}_p) \longrightarrow \mathbb{C}_p$$

- Try to define ??? as an element of

$$\begin{aligned} H_1(\Gamma, \Omega^1_{\mathsf{rig}}(\mathcal{H}_p)^\vee) &= H_1(\Gamma, \mathcal{D}^0_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p))^\vee) \\ &= H_1(\Gamma, \mathcal{C}_{\mathsf{la}}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{C}_p). \end{aligned}$$

- There is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map

$$\mathrm{Div}^0\,\mathcal{H}_p \longrightarrow \mathcal{C}_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{C}_p, \qquad (\tau')-\{\tau\} \mapsto \log_p\left(\frac{z-\tau'}{z-\tau}\right).$$

- We get a map

$$H_1(\Gamma, \mathrm{Div}^0\,\mathcal{H}_p) \longrightarrow H_1(\Gamma, \mathcal{C}_{\mathrm{la}}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{C}_p)$$

- We construct elements of $H_1(\Gamma, \mathrm{Div}^0\,\mathcal{H}_p)$ using real quadratic irrationalities and the description of H_1 in terms of inhomogeneous cycles and boundaries.

Group homology

Let G be a group and M a left G-module.

$$Z_1(G,M) = \mathrm{ker}(\partial : \mathbb{Z}[G] \otimes M \to M),$$
$$\partial(g \otimes m) = gm - m,$$

$$B_1(G,M) = \mathrm{im}(\partial : \mathbb{Z}[G]^{\otimes 2} \otimes M \to \mathbb{Z}[G] \otimes M),$$
$$\partial(g_1 \otimes g_2 \otimes m) = g_2 \otimes g_1^{-1} m - g_1 g_2 \otimes m + g_1 \otimes m,$$

$$H_1(G,M) = Z_1(G,M)/B_1(G,M).$$

1-cycles associated to real quadratic irrationalities

- Let $\mathcal{O} \subset K$ be a real quadratic order such that $(\mathrm{disc}\,\mathcal{O}, Np) = 1$. Assume p is inert in K.

- Let $\psi : \mathcal{O} \longrightarrow R_0^{N^-}(N^+)$ be an optimal embedding.

- There is are two points τ, $\bar{\tau} \in \mathcal{H}_p$, conjugate under $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ such that

$$\mathrm{stab}_\Gamma\,\tau = \mathrm{stab}_\Gamma\,\bar{\tau} = \psi(\mathcal{O}^\times)/\{\pm 1\} \cong \langle \gamma_\psi \rangle.$$

- Thus,

$$\gamma_\psi \otimes \{\tau\} \in Z_1(\Gamma, \mathrm{Div}\,\mathcal{H}_p), \qquad [\gamma_\psi \otimes \{\tau\}] \in H_1(\Gamma, \mathrm{Div}\,\mathcal{H}_p)$$

- But we want elements of $H_1(\Gamma, \mathrm{Div}^0 \mathcal{H}_p)$, not of $H_1(\Gamma, \mathrm{Div}\,\mathcal{H}_p)$!

- Assume that Γ^{ab} is finite and let e be its exponent. (genus zero tame level)

- SES: $0 \longrightarrow \mathrm{Div}^0 \mathcal{H}_p \longrightarrow \mathrm{Div}\,\mathcal{H}_p \longrightarrow \mathbb{Z} \longrightarrow 0.$

- LES:

$$H_2(\Gamma, \mathbb{Z}) \longrightarrow H_1(\Gamma, \mathrm{Div}^0 \mathcal{H}_p)$$
$$\longrightarrow H_1(\Gamma, \mathrm{Div}\,\mathcal{H}_p) \longrightarrow H_1(\Gamma, \mathbb{Z}) = \Gamma^{\mathrm{ab}}$$

- Therefore, $[\gamma_\psi^e \otimes \{\tau\}]$ lifts to an element

$$C_\psi^e \in H_1(\Gamma, \mathrm{Div}^0 \mathcal{H}_p),$$

unique up to Eisenstein classes.

Where were we?

Replace all archimedean objects with their nonarchimedean counterparts:

$$\mathcal{H} \longleftrightarrow \mathcal{H}_p, \quad \Omega^1_{\mathrm{hol}}(\mathcal{H}) \longleftrightarrow \Omega^1_{\mathrm{rig}}(\mathcal{H}_p),$$

$$2\pi i f(\tau) d\tau \longleftrightarrow \ \Phi^{\pm} \in H^1(\Gamma, \Omega^1_{\mathrm{rig}}(\mathcal{H}_p)),$$

$D \in \mathrm{Div}^0 \mathcal{H}$ supported on imaginary quadratic irrationalities

$$\longleftrightarrow \ C_\tau^e \in H_1(\Gamma, \mathrm{Div}^0 \mathcal{H}_p),$$

$\int_D 2\pi i f(\tau) d\tau \in \mathbb{C} \longleftrightarrow \ \langle \Phi^{\pm}, C_\tau^e \rangle \in \mathbb{C}_p$ (Morita duality)

$$\langle \cdot, \cdot \rangle : H^1(\Gamma, \Omega^1_{\mathrm{rig}}(\mathcal{H}_p)) \times H_1(\Gamma, \mathrm{Div}^0 \mathcal{H}_p) \longrightarrow \mathbb{C}_p$$

- Let $q \in p\mathbb{Z}_p$ be the Tate period of $E/\mathbb{Q}_p$ and let $\log_q$ be the branch of the logarithm satisfying $\log_q q = 0$.

- Define $\log : E(\mathbb{C}_p) \to \mathbb{C}_p$ to be given by

$$E(\mathbb{C}_p) \xleftarrow{\text{Tate unif.}} \mathbb{C}_p^\times \xrightarrow{\log_q} \mathbb{C}_p, \qquad \log : E(\mathbb{C}_p) \to \mathbb{C}_p$$

Conjecture. There are points $P_\psi^\pm \in E(H_\mathcal{O}) \otimes \mathbb{Q}$ such that

$$\log P_\psi^\pm = \langle \Phi^\pm, C_\psi^e \rangle.$$

- Such a point P_ψ is called a *Stark-Heegner point*.

Lifting explicitly

- For computational purposes, we need to find an explicit cycle in $Z_1(\Gamma, \mathrm{Div}^0 \mathcal{H}_p)$ mapping to C_ψ^e.

- For $\tau \in \mathcal{H}_p$, and $h, k \in \Gamma$, let

$$
\begin{aligned}
\xi &= hkh^{-1}k^{-1} \otimes \{\tau\} = [h,k] \otimes \{\tau\} \\
&\in \mathbb{Z}[\Gamma] \otimes \mathrm{Div}\, \mathcal{H}_p, \\
\eta &= h^{-1}k^{-1} \otimes (h^{-1}k^{-1}\{\tau\} - \{\tau\}) + k \otimes (h^{-1}\{\tau\} - \{\tau\}) \\
&\quad - h^{-1} \otimes (h^{-1}\{\tau\} - \{\tau\}) - k^{-1} \otimes (k^{-1}\{\tau\} - \{\tau\}) \\
&\in \mathbb{Z}[\Gamma] \otimes \mathrm{Div}^0 \mathcal{H}_p.
\end{aligned}
$$

- Then

$$\xi - \eta = \partial\Big(h \otimes h^{-1} \otimes \{\tau\} + k \otimes k^{-1} \otimes \{\tau\} + 2(1 \otimes 1 \otimes \{\tau\})$$
$$- h \otimes k \otimes \{\tau\} - h^{-1} \otimes k^{-1} \otimes \{\tau\} - hk \otimes h^{-1}k^{-1} \otimes \{\tau\}\Big).$$

- If we can express γ_ψ^e as a product of commutators, then we can use the above formulas to find an element of $Z_1(\Gamma, \mathrm{Div}^0 \mathcal{H}_p)$ representing C_ψ^e.

- Under the assumption that Γ^{ab} is finite, generic group algorithms in Magma will (try to) find generators and relations for $[\Gamma, \Gamma]$ using generators and relations of Γ.

- We solve the word problem in the Fuchsian group $[\Gamma, \Gamma]$ by running Voight's algorithms to compute a fundamental domain for it.

Another interesting case

- Suppose $N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18$, or 25, so that $X_0(N)$ has genus zero but $\Gamma^{\text{ab}} = \Gamma_0(N)^{\text{ab}}$ is infinite.

- LES:

$$H_2(\Gamma, \mathbb{Z}) \longrightarrow H_1(\Gamma, \text{Div}^0 \mathcal{H}_p) \longrightarrow H_1(\Gamma, \text{Div}\, \mathcal{H}_p) \longrightarrow H_1(\Gamma, \mathbb{Z}) = \Gamma^{\text{ab}}$$

- In this case, $H_1(\Gamma, \mathbb{Z})$ is Eisenstein. If $\ell \nmid Np$, then

$$\Big(T_\ell - (\ell+1)\Big)[\gamma_\psi \otimes \{\tau\}] \in \ker \Big(H_1(\Gamma, \text{Div}\, \mathcal{H}_p) \to H_1(\Gamma, \mathbb{Z})\Big),$$

so it lifts to $C_\psi \in H_1(\Gamma, \text{Div}^0 \mathcal{H}_p)$.

- ??? lifting explicitly ???

- Thanks!!

- mgreenbe@ucalgary.ca

- www.math.ucalgary.ca/~mgreenbe