Crystals and Box-Ball systems

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November 21th, 2016

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Definition

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The *Kirillov–Reshetikhin (KR) crystal* $B^{1,s}$ for $U_q'(\widehat{\mathfrak{sl}}_n)$ is

$$B^{1,s} := \left\{ (x_1, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n \mid \sum_{i=1}^n x_i = s \right\}$$

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with *crystal operators*

$$e_i(\ldots, x_i, x_{i+1}, \ldots) = \begin{cases} 0 & \text{if } x_{i+1} = 0, \\ (\ldots, x_i + 1, x_{i+1} - 1, \ldots) & \text{if } x_{i+1} > 0, \end{cases}$$

$$f_i(\ldots, x_i, x_{i+1}, \ldots) = \begin{cases} 0 & \text{if } x_i = 0, \\ (\ldots, x_i - 1, x_{i+1} + 1, \ldots) & \text{if } x_i > 0, \end{cases}$$

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statistics

$$\varepsilon_i(x_1, \ldots, x_n) = x_{i+1} = \max\{k \mid e_i^k(x_1, \ldots, x_n) \neq 0\},$$
$$\varphi_i(x_1, \ldots, x_n) = x_i = \max\{k \mid f_i^k(x_1, \ldots, x_n) \neq 0\},$$

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and *weight function*

$$\mathrm{wt}(x_1, \ldots, x_n) = (x_1, \ldots, x_n) \in \mathbb{Z}^n / \mathbb{Z}(1, \ldots, 1),$$

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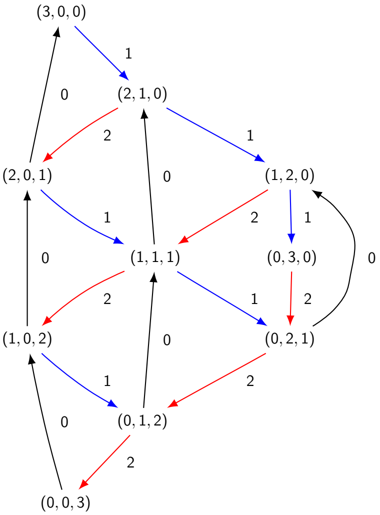
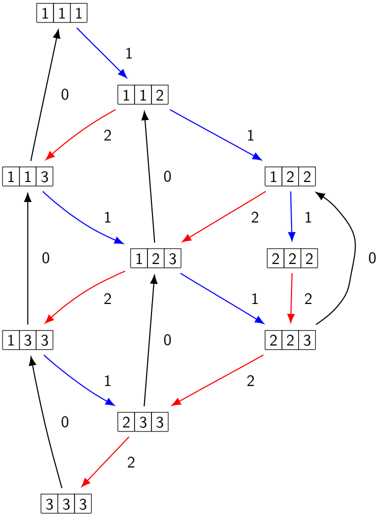
$B^{1,3}$ for $U_q'(\widehat{\mathfrak{sl}}_3)$



$B^{1,3}$ for $U_q'(\widehat{\mathfrak{sl}}_3)$ using tableaux



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Then we have

$$f_i(b) = b_1 \otimes \cdots \otimes b_{j+1} \otimes f(b_j) \otimes b_{j-1} \otimes \cdots \otimes b_L,$$

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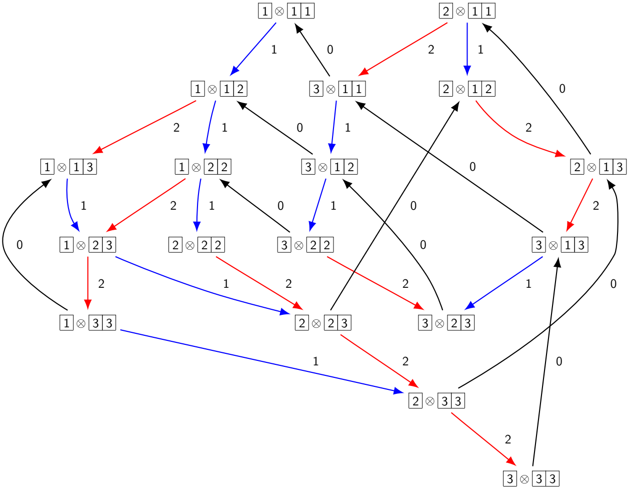
Remark

Our convention is opposite of Kashiwara.

$B^{1,1} \otimes B^{1,2}$ for $U_q'(\widehat{\mathfrak{sl}}_3)$



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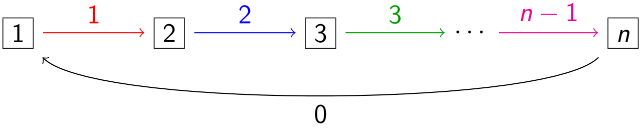

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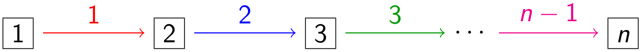

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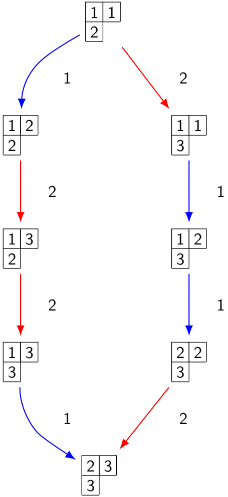
- $U_q(\mathfrak{sl}_n)$-representation V_λ admits crystal basis B_λ, set of all semistandard Young tableaux of shape λ and max entry n.
- Can construct $B_\lambda \subseteq B(1)^{\otimes|\lambda|}$ using tensor product rule and taking the reverse Far-Eastern reading word.

Example B_{21} for $U_q(\mathfrak{sl}_3)$



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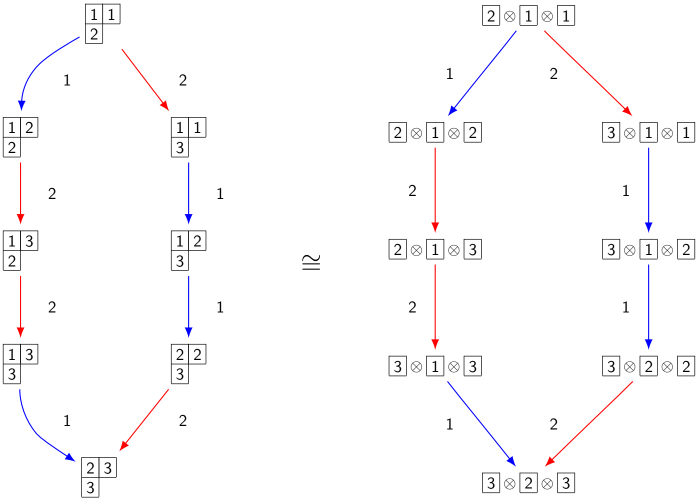

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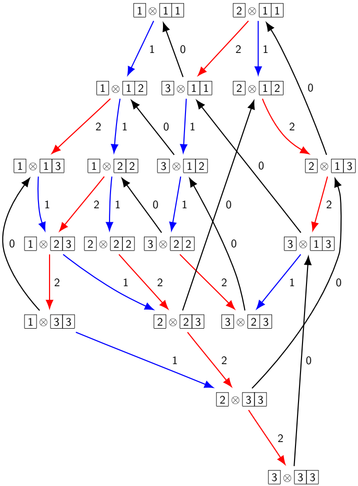
Remark

Generally, the tensor product $\bigotimes_{i=1}^{N} B^{1,s_i}$ is connected.

$B^{1,1} \otimes B^{1,2}$ and $B^{1,2} \otimes B^{1,1}$ for $U_q(\widehat{\mathfrak{sl}}_3)$



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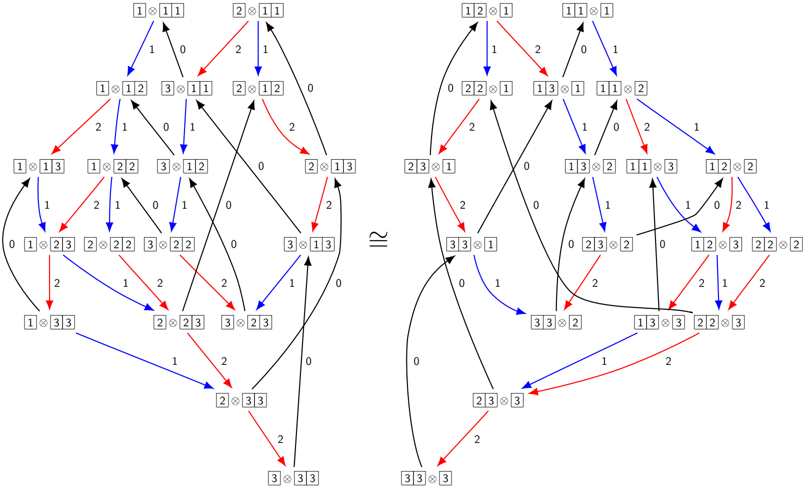

Computing the combinatorial R-matrix

- We have $R(b \otimes b') = \widetilde{b}' \otimes \widetilde{b}$, where $\widetilde{b}' \in B^{1,s'}$ and $\widetilde{b} \in B^{1,s}$ as the unique elements such that

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Avoiding interaction

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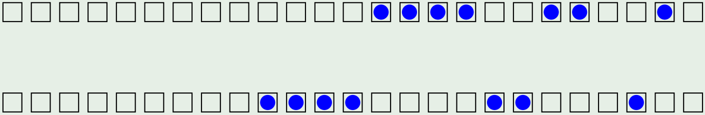

$(t = 0)$

$(t = 1)$

Avoiding interaction

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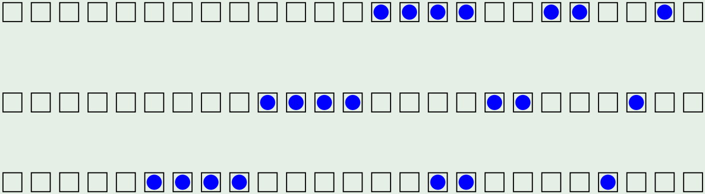

$(t = 0)$

$(t = 1)$

$(t = 2)$

Avoiding interaction

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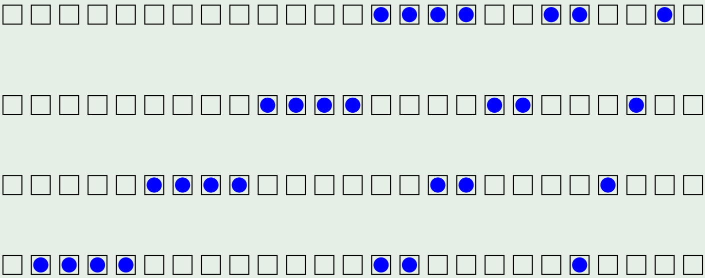

$(t = 0)$

$(t = 1)$

$(t = 2)$

$(t = 3)$

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- After scattering, there is generally a *phase shift*.

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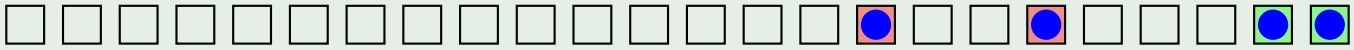

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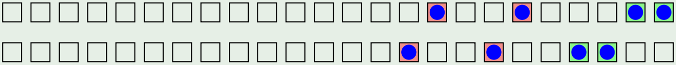

$(t = 0)$

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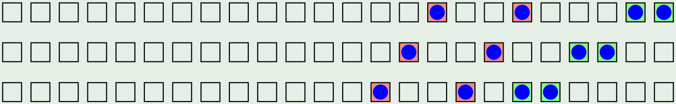

$(t = 0)$

$(t = 1)$

$(t = 2)$

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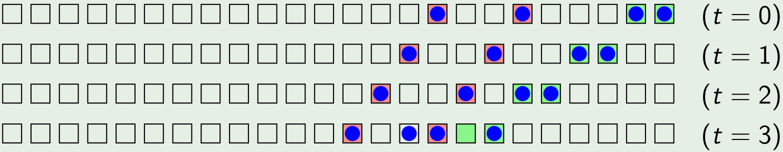

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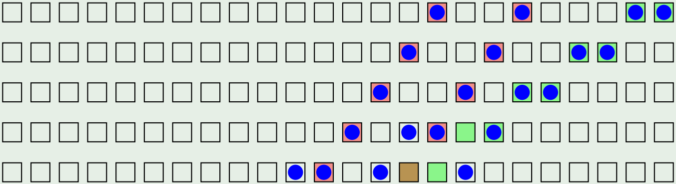

$(t = 0)$

$(t = 1)$

$(t = 2)$

$(t = 3)$

$(t = 4)$

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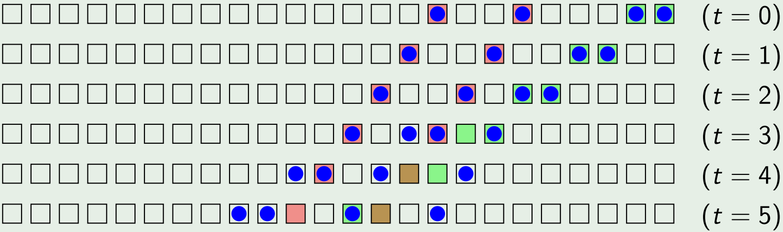

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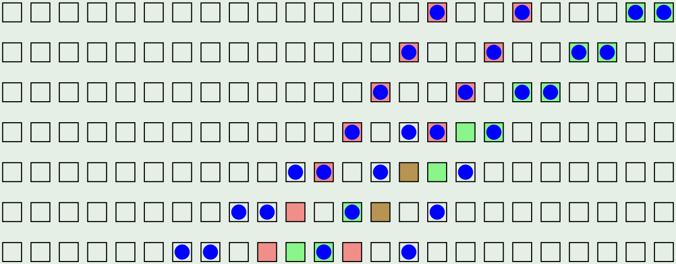

$(t = 0)$
$(t = 1)$
$(t = 2)$
$(t = 3)$
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$(t = 5)$
$(t = 6)$

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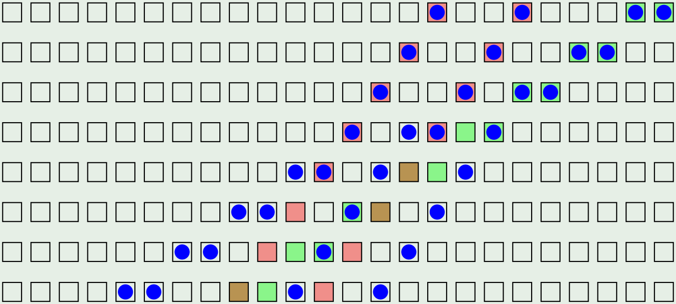

$(t = 0)$
$(t = 1)$
$(t = 2)$
$(t = 3)$
$(t = 4)$
$(t = 5)$
$(t = 6)$
$(t = 7)$

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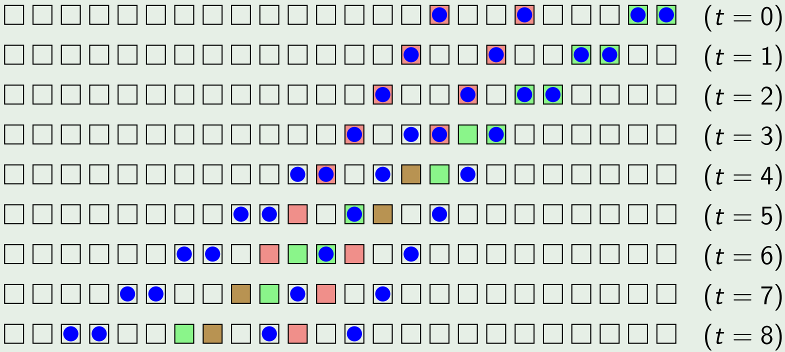

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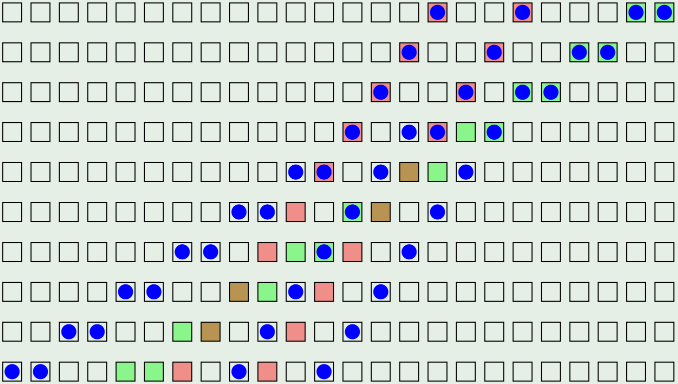

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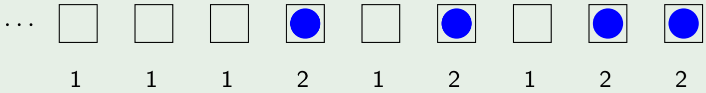

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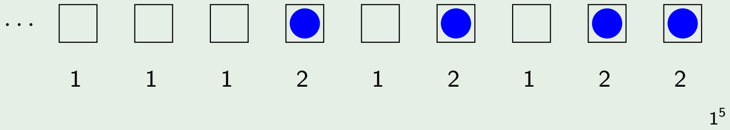

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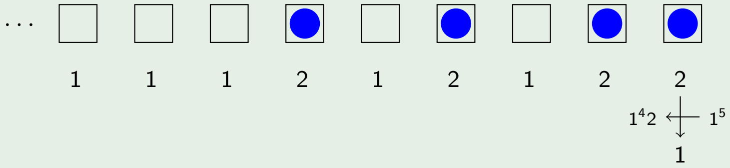

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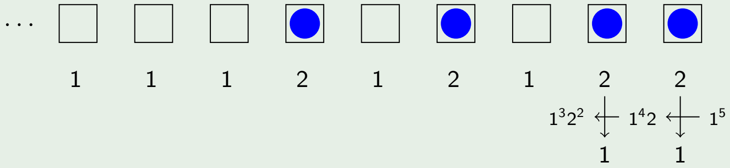

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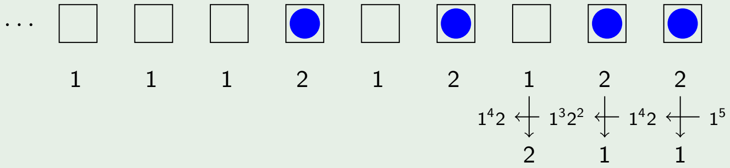

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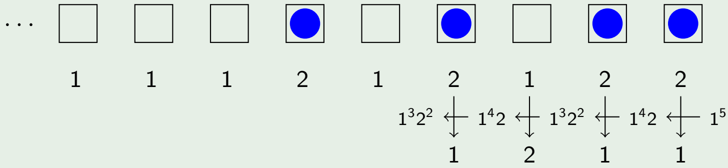

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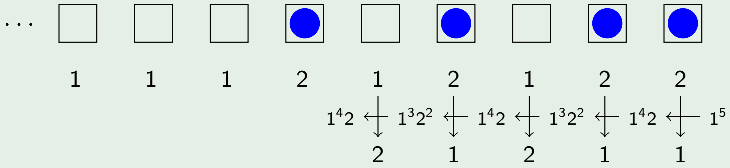

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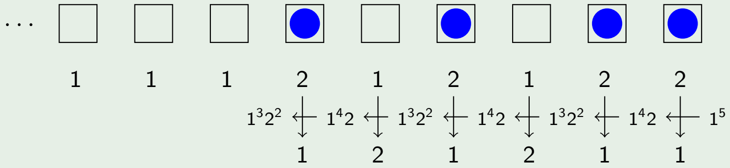

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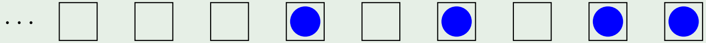

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$$1^4 2 \leftarrow 1^3 2^2 \leftarrow 1^4 2 \leftarrow 1^3 2^2 \leftarrow 1^4 2 \leftarrow 1^3 2^2 \leftarrow 1^4 2 \leftarrow 1^5$$

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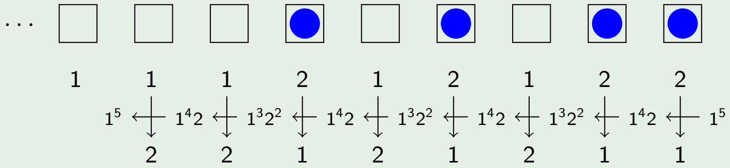

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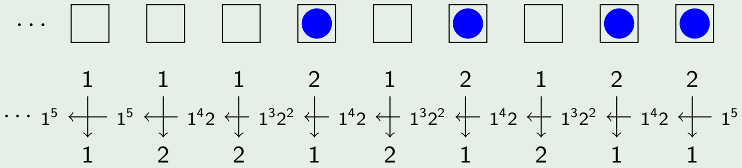

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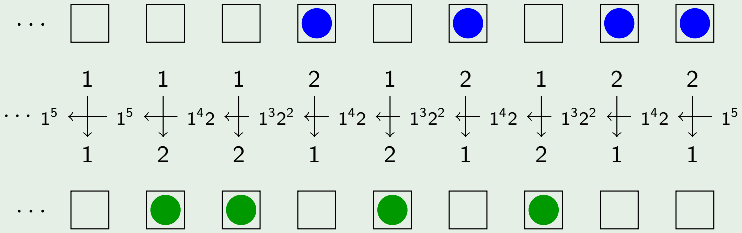

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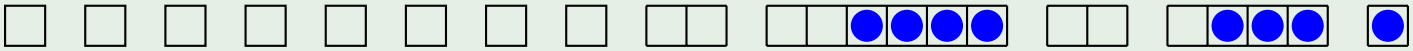

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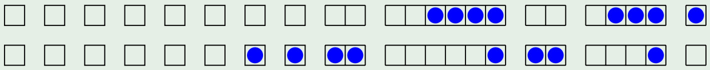

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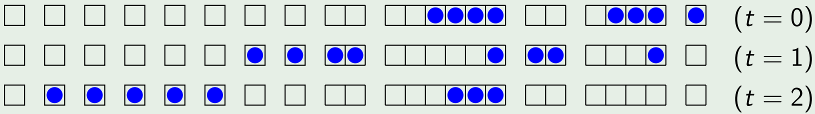

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- Have been well-studied, along with KR crystals, over the past 30 years by numerous authors.

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As usual with partitions, we are allowed to reorder the rows.

Example rigged configurations

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Rigged configurations in $\mathrm{RC}(B^{1,3} \otimes B^{1,5} \otimes B^{1,2} \otimes B^{1,2})$ for $U_q'(\widehat{\mathfrak{sl}}_5)$:

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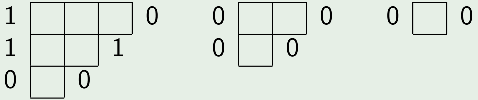

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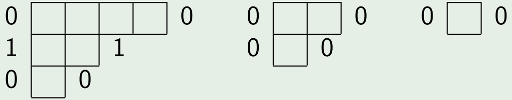

Vacancy numbers are on the left, and riggings are on the right.

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- The combinatorial R-matrix becomes the identity map on rigged configurations under Φ.

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- The inversion bijection Φ^{-1} has been given in terms of the tropicalization of the τ function from the Kadomtsev–Petviashvili (KP) hierarchy.

Thank you!