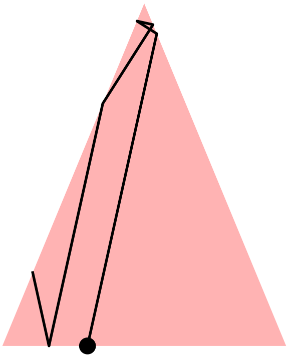
Computing Lyapunov exponents of the Teichmller flow

Vincent Delecroix, Université de Bordeaux (France)

OPEN DREAMKIT

Problem 1: deviation of Birkhoff averages in rational billiards



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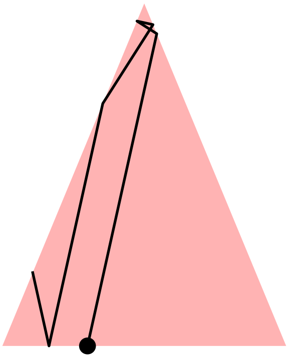
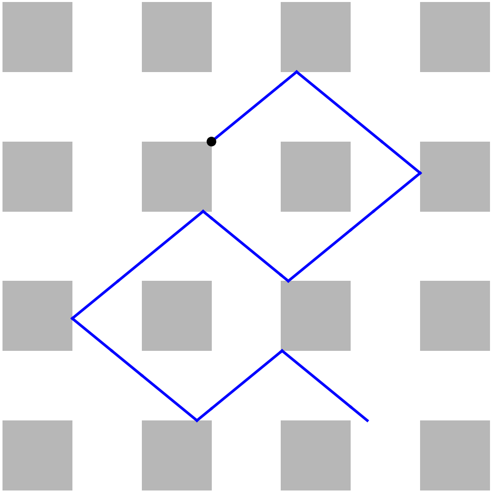


Does the ball touch more often the bottom side or the left side?

Problem 2: Perodic wind-tree model



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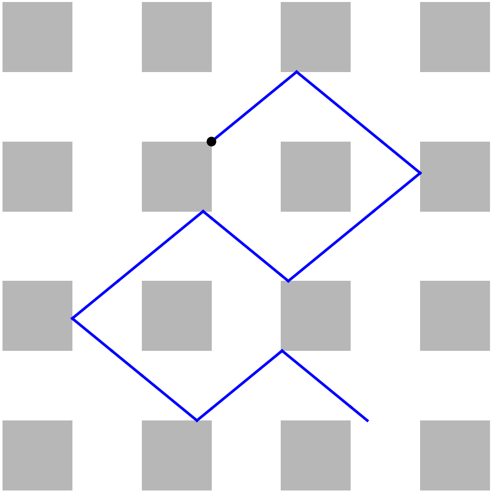
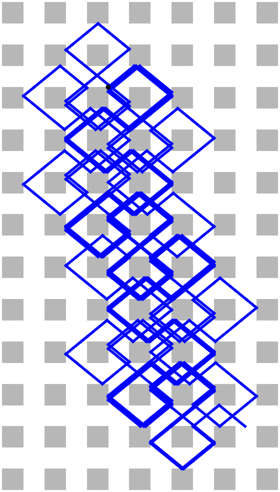


Where does the ball will go?

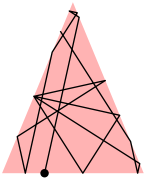
Problem 2: Perodic wind-tree model

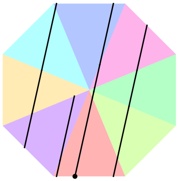


Unfolding rational billiards

. . . jupyter video . . .

First order approximation for Birkhoff sums





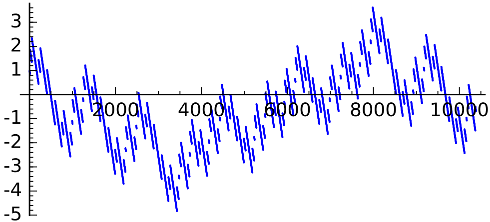
Theorem (Kerckhoff-Masur-Smillie 1985, Veech 1989)

In almost every direction each billiard trajectory equidistributes.

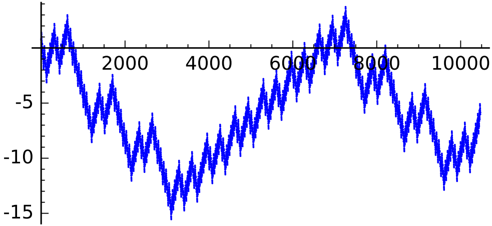
- Up to time T, the number of time a trajectory intersects a given side is equivalent to cT where c is the "transversal length" of the side with respect to the direction.

Second order approximation?

direction $(0.99304, 0.11778)$



direction $(0.23036, 0.9731)$



Deviation of Birkhoff sums

Theorem (Adamczewski, Zorich, Forni, Bufetov, Eskin-Mirzakhani, Chaïka-Eskin, Delecroix-Hubert-Lelièvre, . . .)

Let S be a translation surface and γ a segment in S. There exists a number λ so that for almost every direction θ and all $x \in S$ we have

$$\sup_{0 \le t \le T} \log |i(S, \gamma, \theta, x, t) - t \mu_\theta(\gamma)| \sim \lambda \log(T)$$

where

- $i(S, \gamma, \theta, x, t)$ *is the number of intersection of* γ *with the trajectory of length* t *issued from* x *in direction* θ*,*
- $\mu_\theta(\gamma)$ *is the transverse measure of* γ *in direction* θ*.*

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The number λ is a Lyapunov exponent of the Kontsevich-Zorich cocycle!

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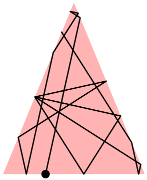
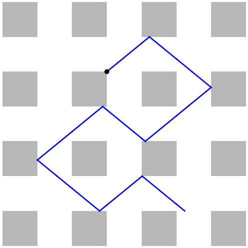
The number λ is a Lyapunov exponent of the Kontsevich-Zorich cocycle! Theorems and/or SageMath can sometimes compute λ!

Deviation of Birkhoff sums



$\lambda = 1/3$
(Bainbridge, McMullen)



$\lambda = 2/3$
(Delecroix-Hubert-Lelièvre)

Teichmüller flow and Kontsevich-Zorich cocycle

(the slide without picture)

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4. The Kontsevich-Zorich cocycle is basically the differential Dg_t of the Teichmüller flow.
5. There is a magic formula to compute the sum of positive exponents (Eskin-Kontsevich-Zorich).

Computations in SageMath

Using Sage to compute these Lyapunov exponents. . .