Simplicial sets in Sage

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Simplicial sets

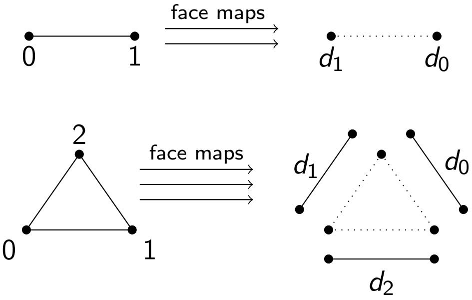
A *simplicial set* is a combinatorial model for a topological space, consisting of

- a set X_n of n-simplices for each integer $n \geq 0$, and
- face maps d_i and degeneracy maps s_j between the sets X_n.

The maps have to satisfy certain "obvious" identities.

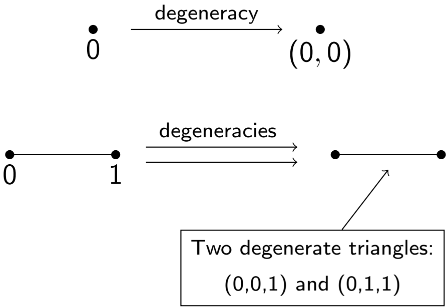
Face maps

Each n-simplex has $n+1$ faces, so there are $n+1$ face maps
$d_i : X_n \to X_{n-1}, \quad 0 \leq i \leq n.$



Degeneracy maps

An n-simplex determines $n+1$ degenerate $(n+1)$-simplices, so there are $n+1$ degeneracy maps: $s_j : X_n \to X_{n+1}$, $0 \leq j \leq n$.



A morphism of simplicial sets $X \to Y$ is the obvious thing: a collection of maps $X_n \to Y_n$ which are compatible with the face and degeneracy maps.

Examples

Example

The simplest examples are

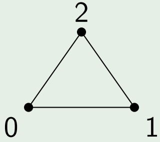
- the empty simplicial set: $X_n = \emptyset$ for all n
- a point: $X_n = \text{(singleton)}$ for all n

Examples

Any simplicial complex can be made into a simplicial set: just add degeneracies freely.

Example

Triangulation of S^1:



$$X_0 = \{0, 1, 2\}$$
$$X_1 = \{01, 02, 12, s_0(0), s_0(1), s_1(2)\}$$
$$X_2 = \{s_0(01), s_1(01), s_0(02), s_1(02), s_0(12), s_1(12), s_1 s_0(0), s_1 s_0(1), s_1 s_0(2)\}$$
$$X_3 = \{\ldots \text{degenerate simplices} \ldots\}$$
$$\vdots$$

Examples

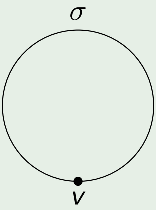
The n-sphere, more efficiently:
One vertex v and one nondegenerate n-simplex σ (plus many degenerate simplices).

Example

$n = 1$:

$X_0 = \{v\} \quad X_1 = \{\sigma, s_0 v\} \quad X_2 = \{s_0\sigma, s_1\sigma, s_1 s_0 v\} \quad \cdots$

Face maps: $d_0\sigma = d_1\sigma = v$. The others are automatic.

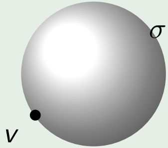


Examples

The n-sphere, more efficiently:
One vertex v and one nondegenerate n-simplex σ (plus many degenerate simplices).

Example

$n = 2$:
$X_0 = \{v\}$, $X_1 = \{s_0 v\}$, $X_2 = \{\sigma, s_1 s_0 v\}$, ...
Face maps: $d_i \sigma = s_0 v$ for all i. The others are automatic.



Motivation: size

As you can see from the sphere examples, simplicial sets can be more efficient than simplicial complexes if you ignore degenerate simplices.

Fortunately, you can frequently ignore them.

Motivation: Homotopy theory

From an abstract point of view, you can study homotopy theory purely using the category of simplicial sets.

- The homotopy theory of simplicial sets is equivalent to the homotopy theory of topological spaces.
- For some homotopy theorists, “space” means “simplicial set”.
- Some constructions are easier for simplicial sets, some for topological spaces.
- Some constructions are easier for simplicial sets, some for simplicial complexes.

Motivation: products

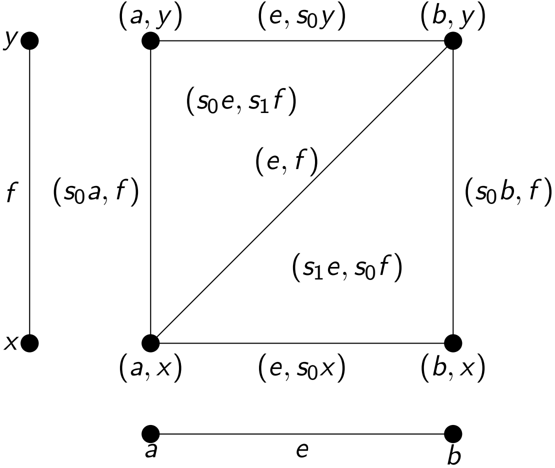
Simplicial complexes can be annoying to work with. For example, products: how do you triangulate the product of two simplices? (Known and understood, but requires a little work.)

Example

Taken from Sage documentation: if T, K are minimal triangulations of the torus and Klein bottle, with 14 and 16 facets respectively, then $T \times K$ has 1344 facets and takes a little time to compute.

Products of simplicial sets: if X and Y are simplicial sets, then their product has n-simplices $X_n \times Y_n$.

Products of simplicial sets: if X and Y are simplicial sets, then their product has n-simplices $X_n \times Y_n$. For example, if X and Y are both 1-simplices:



Motivation: Nerves

The nerve of a category (or group or monoid) is naturally constructed as a simplicial set.

The *nerve* (= *classifying space*) of a group:

- one vertex
- one edge for each element of the group
- one n-simplex for each n-tuple $(a_1, a_2, \cdots, a_n)$, non-degenerate if $a_i \neq 1$ for all i
- face maps: multiply consecutive elements
- degeneracy maps: insert 1

Nerves

A little more detail:

- $d_0(a_1 a_2 \cdots a_n) = a_2 \cdots a_n$
- $d_n(a_1 a_2 \cdots a_n) = a_1 \cdots a_{n-1}$
- $d_i(a_1 a_2 \cdots a_n) = a_1 \cdots (a_i a_{i+1}) \cdots a_{n-1}$, $1 \leq i \leq n-1$
- $s_j(a_1 \cdots a_n) = a_1 \cdots a_j 1 a_{j+1} \cdots a_n$, $0 \leq j \leq n$ (insert 1 in the jth spot).

Nerves

Similar for monoids or categories: one vertex for each object, one 1-simplex for each morphism, one n-simplex for each collection of n composable morphisms.

Also, given the nerve of a category, you can recover the category.

Question

Are categories in good enough shape in Sage to be able to define the nerve of a (finite) category?

Example

Real projective space. $\mathbf{R}P^\infty$ is the classifying space of the group $\mathbf{Z}/2\mathbf{Z}$. There is one non-degenerate simplex in each dimension. $\mathbf{R}P^n$ is its n-skeleton.
Look at the f-vectors for *simplicial complex* versions of $\mathbf{R}P^n$ for small values of n:

$\mathbf{R}P^2$: $(6, 15, 10)$
$\mathbf{R}P^3$: $(11, 51, 80, 40)$
$\mathbf{R}P^4$: $(16, 120, 330, 375, 150)$
$\mathbf{R}P^5$: $(63, 903, 4200, 8400, 7560, 2520)$

In comparison, as a *simplicial set*:
$\mathbf{R}P^n$: $(1, 1, 1, 1, \cdots, 1)$

As a result, computing is much faster with the simplicial set model.

Example

Classifying space of C_3. In Sage, the group C_3 (constructed via `C3 = groups.misc.MultiplicativeAbelian([3])`) has three elements, 1, f, f^2. So its classifying space has two nondegenerate 1-simplices, four non-degenerate 2-simplices ($f * f$, $f^2 * f$, $f * f^2$, $f^2 * f^2$), eight non-degenerate 3-simplices, etc.

Example

Classifying space of Σ_3 and its fundamental group.

Example

Complex projective space. $\mathbf{C}P^n$ is the $2n$-skeleton of the classifying space of the Lie group S^1. Sage can't construct it that way, but work of Sergeraert (*Kenzo*, *CAT*) leads to constructions we can use in Sage.

f-vectors as *simplicial complexes*:

$\mathbf{C}P^2$: $(9, 36, 84, 90, 36)$
$\mathbf{C}P^3$: not implemented
$\mathbf{C}P^4$: not implemented

As *simplicial sets*:

$\mathbf{C}P^2$: $(1, 0, 2, 3, 3)$
$\mathbf{C}P^3$: $(1, 0, 3, 10, 25, 30, 15)$
$\mathbf{C}P^4$: $(1, 0, 4, 22, 97, 255, 390, 315, 105)$

To do:

- good conversions from simplicial complexes (and other objects) to simplicial sets
- simplicial abelian groups, k-skeleton of $K(\pi, n)$
- infinite simplicial sets
- general framework for simplicial objects in a category
- higher homotopy groups (?)