SnapPy - Part I

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SnapPy is developed by Marc Culler, Nathan Dunfield, Matthias Goerner, and Jeff Weeks.

With contributions from Mark Bell, Tracy Hall, Saul Schleimer, Malik Obeidin, Robert Lipschitz, Jennet Dickinson and many others...

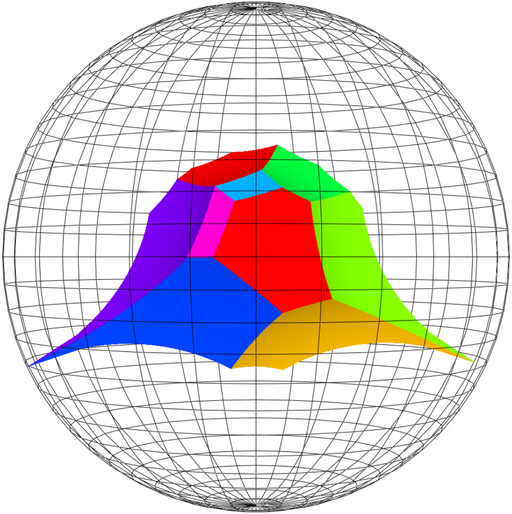
Hyperbolic 3-space $\mathbb{H}^3$ is the unique homogeneous Riemannian 3-manifold with all sectional curvatures equal to -1. Alternatively, $\mathbb{H}^3 = PSL_2(\mathbb{C})/SO(3)$ with a normalized invariant metric.

There is a natural compactification $\overline{\mathbb{H}}^3 \doteq \mathbb{H}^3 \sqcup S^2_\infty$ which is homeomorphic to a 3-ball. The boundary S^2_∞ is the *sphere at infinity*. It has a canonical conformal structure.

There are 8 simply-connected Riemannian 3-manifolds whose full isometry group acts transitively with compact point stabilizers, and which admit a compact quotient under the action of some discrete torsion free group of isometries. These are the "geometries" in Thurston's Geometrization Conjecture, proved by Grisha Perelman in 2003.

- $\mathbb{H}^3$ is the open unit ball in $\mathbb{R}^3$;
- S^2_∞ is the unit sphere;
- A line is an arc of a circle which is perpendicular to S^2_∞;
- A plane is the intersection of the unit ball with a Euclidean sphere perpendicular to S^2_∞;
- The metric is conformal to the Euclidean metric.

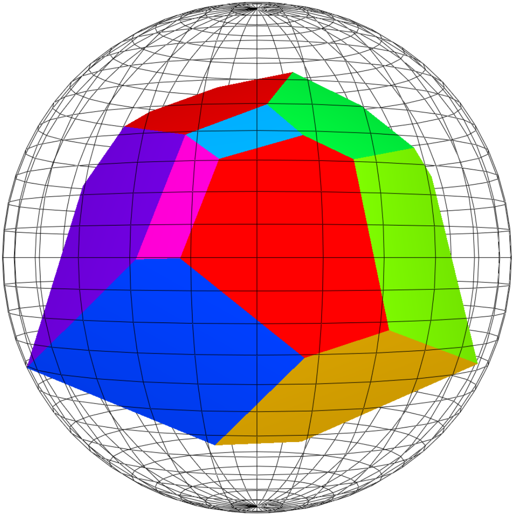
A convex polyhedron in the Poincaré model:



Hyperbolic 3-Space: *The Klein model*

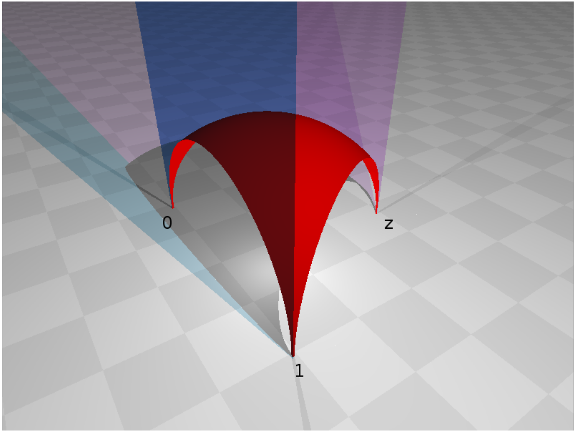
- $\mathbb{H}^3$ is the open unit ball in $\mathbb{R}^3$;
- S^2_∞ is the unit sphere;
- A line is the intersection of the ball with a Euclidean line.
- A plane is the intersection of the ball with a Euclidean plane.
- The metric is *not* conformal to the Euclidean metric.

A convex polyhedron in the Klein model:



- $\mathbb{H}^3 = \{(x, y, t) | t > 0\} \subset \mathbb{R}^3$;
- S^2_∞ is the plane $t = 0$ together with the point ∞. If the plane is identified with $\mathbb{C}$, isometries of $\mathbb{H}^3$ extend to M0bius transformations on $\mathbb{C}$ ($z \to \frac{az+b}{cz+d}$);
- A line is a circle orthogonal to $t = 0$, or a vertical line.
- A plane is a half-sphere with center on $t = 0$, or a vertical plane.
- The metric is the Euclidean metric scaled by $\frac{1}{t}$.

An ideal tetrahedron in the upper half-space:



This tetrahedron is the convex hull of the points 0, ∞, 1, z. Up to isometry, every ideal tetrahedron has this form.

The closure in $\overline{\mathbb{H}}^3$ of a plane in $\mathbb{H}^3$ is a 2-disk whose boundary is a geometric circle on S^2_∞. Call this an extended plane.

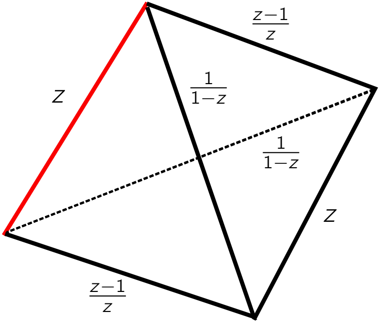
Each extended plane in $\overline{\mathbb{H}}^3$ divides $\overline{\mathbb{H}}$ into two closed half-spaces which meet along the extended plane. The *convex hull* of a subset X of $\overline{\mathbb{H}}^3$ is the intersection of all half-spaces containing X.

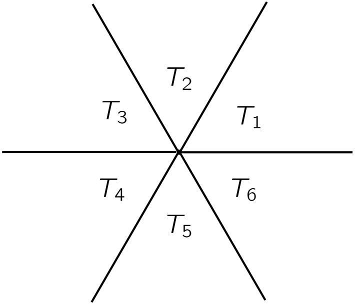
An *ideal tetrahedron* is the convex hull of four points on S^2_∞. Every ideal tetrahedron has three symmetries which interchange two pairs of opposite edges and reflect the other two edges.

The cross-ratio of the vertices of an *ordered* ideal tetrahedron is invariant under *order-preserving* isometries. Take an ordered tetrahedron Δ with cross-ratio z and apply an order 2 symmetry. The new ordered tetrahedron will have cross-ratio z, $\frac{1}{1-z}$, or $\frac{z-1}{z}$.

Remark: The boundary of the convex hull of any closed subset of S^2_∞ is a *pleated surface* bent along a *geodesic lamination*.

The *shape* of an ideal tetrahedron assigns complex numbers to the edges in the following pattern.





If an ordered tetrahedron $\Delta = (v_0, v_1, v_2, v_3)$ is translated to $(0, 1, \infty, z)$, then the shape assigns z to the edge $v_0 - v1$. When $\text{Im}(z) > 0$ say that Δ is *positively oriented*.

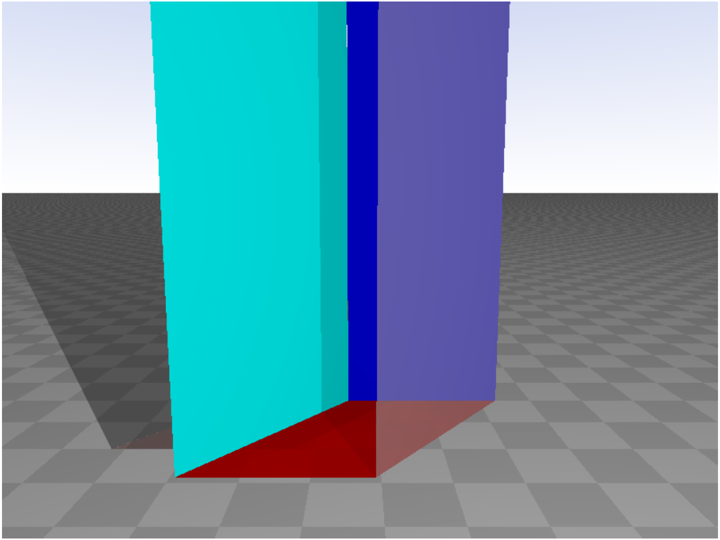
If tetrahedra $T_1, T_2, \ldots, T_n$ fit around an edge as shown, and the shape values for that edge are $z_1, z_2, \ldots, z_n$, then $z_1 \cdots z_n = 1$.

Consider a co-compact discrete group Γ of Euclidean translations of $\mathbb{R}^2$ with fundamental domain a parallelogram P. The action extends to an action by translations on $\mathbb{R}^3$, which are isometries of the upper half-space model.

The region $D = P \times [1, \infty)$ is a fundamental domain for the action of Γ on $\mathbb{R}^2 \times [1, \infty)$. The quotient is diffeomorphic to $T^2 \times [1, \infty)$ and inherits a complete hyperbolic metric. The hyperbolic manifold, D/Γ is called a *cusp neighborhood*. A simple computation, using $dV_{\text{Hyp}} = \frac{1}{t^3} dV_{\text{Euc}}$, shows that cusp neighborhoods have finite volume.

A 3-manifold M with a complete hyperbolic metric of finite volume is either compact or has finitely many ends, each isometric to a cusp neighborhood. Thus M is diffeomorphic to the interior of a compact 3-manifold with torus boundary components.

A cusp neighborhood.



A 3-manifold M has a *geometric structure* if

- M arises as the quotient of one of the 8 geometries by a discrete torsion-free group of isometries.
- The induced Riemannian metric on M is complete.

Every 3-manifold is a connected sum of *irreducible* 3-manifolds which cannot be decomposed as a non-trivial connected sum.

The Geometrization Conjecture asserts that every compact irreducible 3-manifold with (possibly empty) torus boundary can be cut along a canonical family of tori to produce (open) geometric manifolds.

For the 7 non-hyperbolic geometries, the geometric manifolds have been classified.

A finite volume hyperbolic 3-manifold has a *unique* hyberbolic structure by Mostow-Prasad rigidity. So geometric invariants are topological invariants. (!!!!)

SnapPy's data structure for representing a 3-manifold M contains:

An *ideal triangulation*:

- A pseudo-manifold K constructed by identifying pairs of faces of finitely many tetrahedra so that every vertex link is either a sphere or a torus. (To recover M, take the realization and delete the vertices with torus links.)
- A complex shape parameter assigned to one edge of each tetrahedron. (The parameters for the other edges are determined.)

The shapes satisfy the *gluing equations*:

- For each equivalence class e of edges, the product of the shapes assigned to the edges in e equals $e^{2\pi i}$.

These equations have the simple form:

$$z_1^{a_1}(1-z_1)^{b_1}\cdots z_k^{a_k}(1-z_k)^{b_k} = c = \pm 1$$

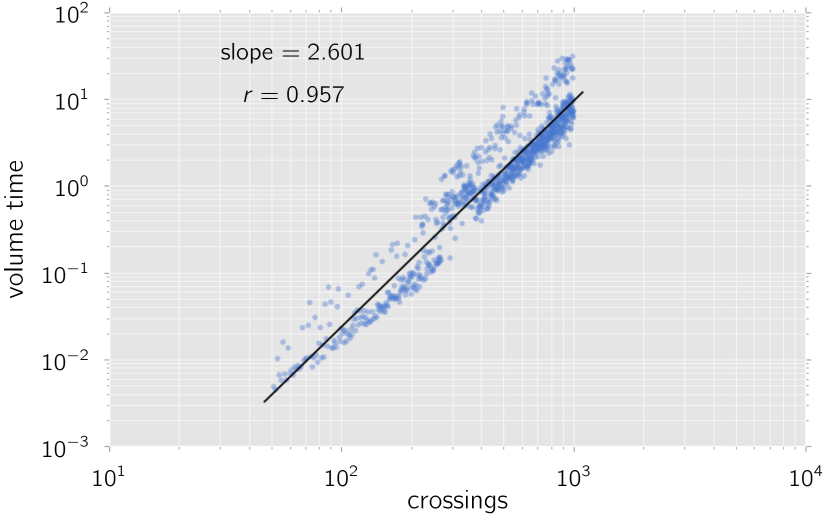
The complex dimension of the space of solutions to the gluing equations is equal to the number of cusps. Adding a "completeness equation" for each cusp cuts the dimension down to 0, giving finitely many solutions.

In practice, if the manifold has a hyperbolic structure then it will be described by one of these finitely many solutions. This means that $M = \mathbb{H}^3/\Gamma$ and there is a Γ-invariant tiling of $\mathbb{H}^3$ by tetrahedra of the specified shapes.

In practice, starting with all regular ideal tetrahedra, Newton's method converges extremely quickly to the hyperbolic structure (if one exists, as almost always happens).

Here is a plot of times required for computing hyperbolic structures on knot complements, against the number of crossings in the knot diagram.

Time to compute volumes of knot complements



Without the completeness equations, solutions to the gluing equations near the hyperbolic structure correspond to *incomplete* hyperbolic structures.

Under certain conditions, the completion of one of these incomplete metrics adds exactly a circle of points at the end of a cusp, producing a hyperbolic manifold with different topology.

The topological operation, called *Dehn filling*, is: add a boundary torus and then glue $S^1 \times D^2$ to the new boundary component. The topology of the result is determined by the homotopy class of the curve $\{*\} \times \partial D^2$ in the torus.

All but finitely many dehn fillings of a cusp yield a hyperbolic manifold. Finding the hyperbolic structure involves adding a *filling equation* instead of the completeness equation for the cusp. Again, *in practice* Newton's method is amazingly effective.

Demo

Code profile

Under the hood, SnapPy is:

- C kernel: 40K LOC started by Weeks in 1990.
- Cython wrapper: 5K LOC.
- Python code: 20K LOC.
- Databases with 2M+ manifolds.
- Modules hosted on PyPI.
- User-friendly native installers provided for OS X and Windows.

Why we're here: SnapPy and SageMath are friends! Nathan will expand on this after the break . . .