Simplicial Complexes (And Sage, Of Course!) From A Combinatorialist's Point Of View

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Simplicial Complexes

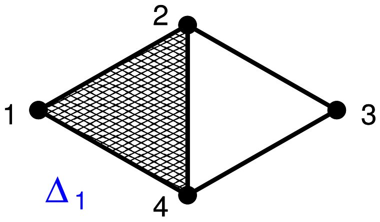
Let V be a finite set of vertices. Typically $V = [n] = \{1, 2, \ldots, n\}$.

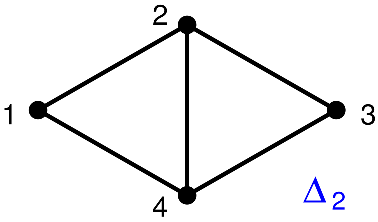
Definition

An (abstract) simplicial complex Δ on V is a family of sets $\Delta \subseteq 2^V$ such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ are called faces or simplices.

- Δ gives rise to a topological space (e.g., its standard geometric realization in $\mathbb{R}^n$)
- $\dim \sigma = |\sigma| - 1$; $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$
- Face poset $F(\Delta)$: set of all faces, partially ordered by inclusion
- f-vector: enumerates faces by dimension

Simplicial Complexes





$\Delta_1 = \langle 124, 23, 34 \rangle$

$\dim \Delta_1 = 2$

$f(\Delta_1) = (1, 4, 5, 1)$

not pure

$\Delta_2 = \langle 12, 14, 23, 24, 34 \rangle$

$\dim \Delta_2 = 1$

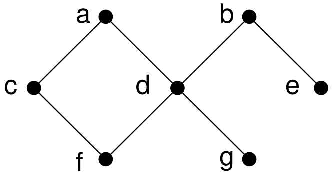
$f(\Delta_2) = (1, 4, 5)$

pure

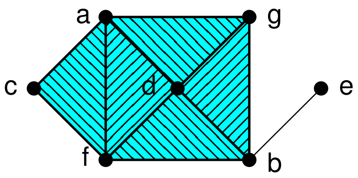
1. Order complexes of posets.

The order complex $\Delta(P)$ of a finite poset P is the simplicial complex on P whose faces are its chains, i.e., the sets

$$\{C \subseteq P \mid \text{ every two elements of } C \text{ are comparable.}\}$$



P



Δ(P)

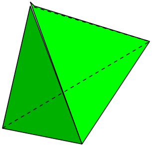
1. Order complexes of posets.

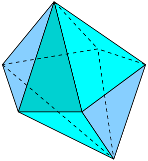
- Typical posets you'd want to do this with: lattice of flats of a matroid, Bruhat order or weak order on a Coxeter group, ...
- Often, combinatorics of $P \iff$ topology of $\Delta(P)$
- Note: $\Delta(F(\Delta))$ is the barycentric subdivision of Δ.

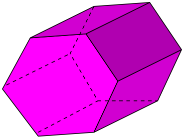
2. Polytopes.

A polytope P is the convex hull of a finite set of n points in $\mathbb{R}^d$.

If $n > d$ and the points are chosen generically, then ∂P is a simplicial complex.







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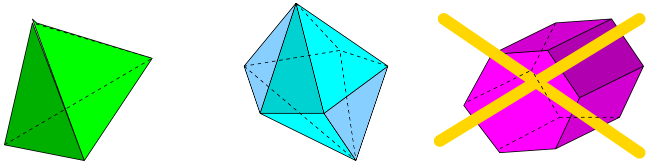

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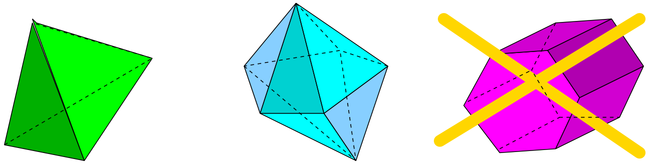


Problem: What are the possible f-vectors of simplicial polytopes?

3. Stanley-Reisner theory.

Let $\Bbbk$ be a base ring (typically $\mathbb{Z}$ or a field), $R = \Bbbk[x_1, \ldots, x_n]$, and $I \subseteq R$ an ideal generated by square-free monomials.

Definition

The Stanley-Reisner complex of I is

$$\Delta(I) = \left\{ \sigma \subseteq [n] : \ \prod_{j \in \sigma} x_j \notin I \right\}.$$

SR ideals also arise from simplicial fans (as in Volker's talk).

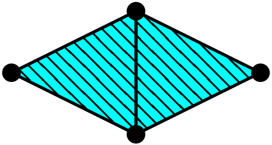
Conversely, every complex Δ on $[n]$ has a Stanley-Reisner ring

$$\Bbbk[\Delta] = R \ / \ \left\langle \prod_{j \in \sigma} x_j : \ \sigma \text{ a minimal nonface of } \Delta \right\rangle.$$

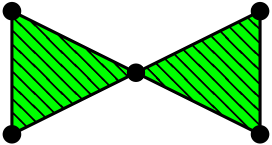
Definition

Suppose $\dim \Delta = d - 1$ and $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$.
The *h-vector* $h(\Delta) = (h_0, \ldots, h_d)$ is given by

$$h_k = \sum_{i=0}^{k} \binom{d-i}{k-i} (-1)^{k-i} f_{i-1}.$$



f = (1, 4, 5, 2)
h = (1, 1)



f = (1, 5, 6, 2)
h = (1, 2, **–1**)

The h-Vector

- Knowing $f(\Delta)$ is equivalent to knowing $h(\Delta)$.
- The Stanley-Reisner ring $S = \mathbb{k}[\Delta]$ is graded:

$$S = \bigoplus_{i \geq 0} S_i = \bigoplus_{i \geq 0} \mathbb{k}\langle \text{monomials supported on a face of } \Delta \rangle$$

with Hilbert series

$$\sum_{i \geq 0} (\dim_{\mathbb{k}} S_i) q^i = \frac{h_0 + h_1 q + \cdots + h_d q^d}{(1-q)^d}.$$

Example

If $P \subseteq \mathbb{R}^d$ is a simplicial polytope, then $h(\partial P)$ is strictly positive and satisfies $h_i = h_{d-i}$ (the Dehn-Sommerville equations).

Problem

What (if anything) do the h-numbers count?

Shellable Complexes

Definition

A simplicial complex Δ is shellable if its facets can be ordered $F_1, \ldots, F_k$ such that for every $i > 1$, the face set

$$\langle F_i \rangle \setminus \langle F_1, \ldots, F_{i-1} \rangle$$

has a unique minimum element R_i.

Equivalently, the subcomplex of F_i along which it is attached, i.e.,

$$\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle,$$

is a pure, codimension-1 subcomplex of F_i.

Shellable Complexes

Shellability is a strong property!

Topologically, shellable complexes are wedges of spheres.

Combinatorially, shellability provides a combinatorial interpretation of the h-vector. In the simplest case where Δ is pure, it is

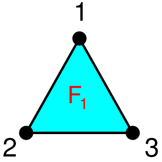
$$h_j = \#\{i : \ |R_i| = j\}$$

and h_d is the number of spheres in the wedge.

Theorem (Bruggesser–Mani)

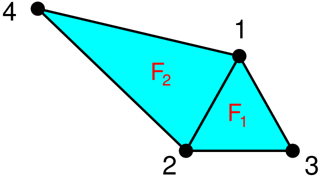
Convex simplicial spheres are shellable.

Example: The Octahedron



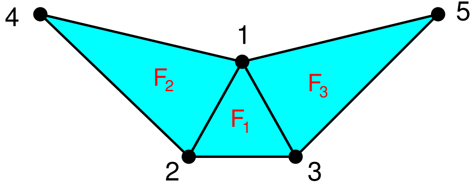
$[\varnothing, 123]$

Example: The Octahedron



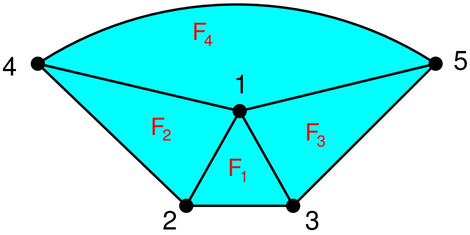
$[\varnothing, 123]$
$\cup\ [4, 124]$

Example: The Octahedron



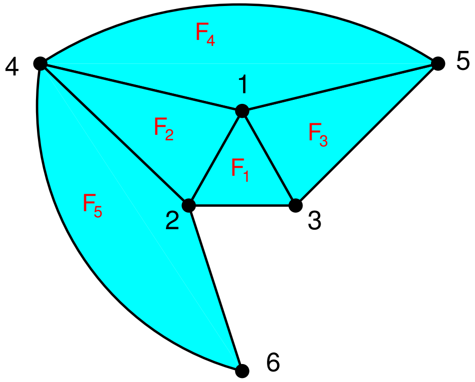
$[\varnothing, 123]$
$\cup\ [4, 124]$
$\cup\ [5, 135]$

Example: The Octahedron



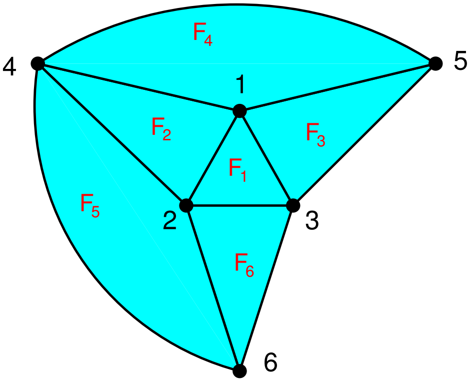
$[\varnothing, 123]$
$\cup\ [4, 124]$
$\cup\ [5, 135]$
$\cup\ [45, 145]$

Example: The Octahedron



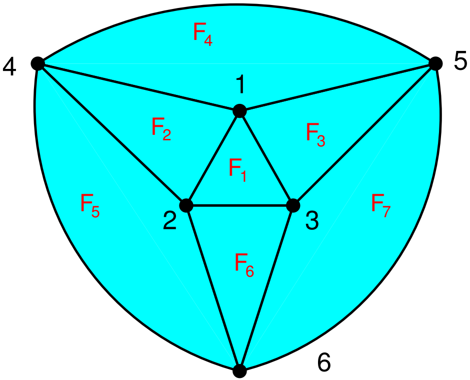
$[\varnothing, 123]$
$\cup\ [4, 124]$
$\cup\ [5, 135]$
$\cup\ [45, 145]$
$\cup\ [6, 246]$

Example: The Octahedron



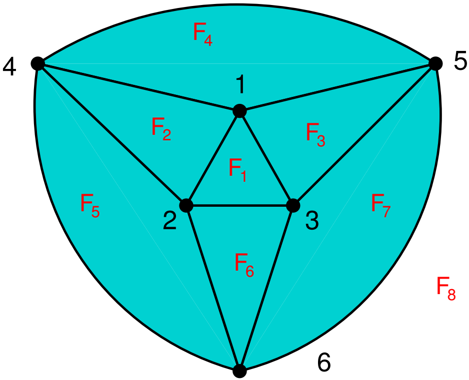
$[\varnothing, 123]$
$\cup\ [4, 124]$
$\cup\ [5, 135]$
$\cup\ [45, 145]$
$\cup\ [6, 246]$
$\cup\ [36, 236]$

Example: The Octahedron



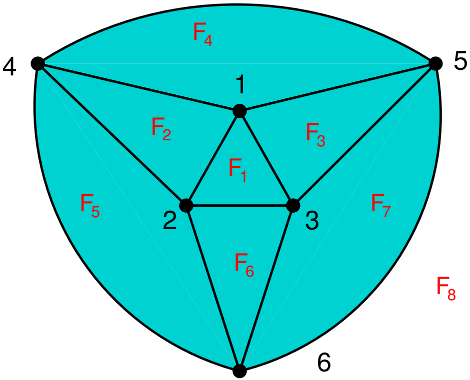
$[\varnothing, 123]$
$\cup\ [4, 124]$
$\cup\ [5, 135]$
$\cup\ [45, 145]$
$\cup\ [6, 246]$
$\cup\ [36, 236]$
$\cup\ [56, 356]$

Example: The Octahedron



$[\varnothing, 123]$
$\cup\ [4, 124]$
$\cup\ [5, 135]$
$\cup\ [45, 145]$
$\cup\ [6, 246]$
$\cup\ [36, 236]$
$\cup\ [56, 356]$
$\cup\ [456, 456]$

Example: The Octahedron



$[\emptyset, 123]$
$\cup\ [4, 124]$
$\cup\ [5, 135]$
$\cup\ [45, 145]$
$\cup\ [6, 246]$
$\cup\ [36, 236]$
$\cup\ [56, 356]$
$\cup\ [456, 456]$

So $h(\Delta) = (1,\ 3,\ 3,\ 1)$.

Definition

Let $\Bbbk$ be a ring. An $\Bbbk$-algebra A is Cohen-Macaulay (CM) over $\Bbbk$ iff its Krull dimension equals its depth as an $\Bbbk$-module. A simplicial complex is Cohen-Macaulay iff its Stanley-Reisner ring is CM.

Theorem (Hochster 1972, Reisner 1976)

Δ is Cohen-Macaulay over $\Bbbk$ iff for every $\sigma \in \Delta$,

$$\tilde{H}_j(\text{link}_\Delta(\sigma); \Bbbk) = 0 \qquad \forall j < \dim \Delta - \dim \sigma - 1.$$

where

$$\text{link}_\Delta(\sigma) = \{\tau \in \Delta : \ \tau \cap \sigma = \varnothing \text{ and } \tau \cup \sigma \in \Delta\}.$$

Cohen-Macaulay Complexes

- Shellable implies CM over any base ring,
- Reisner's theorem means that checking CMness reduces to computing links and homology — exponential, but easier than shellability,
- Δ CM $\implies$ $h(\Delta)$ nonnegative and gap-free.
- Constraints on CM h-vectors are the same as those on shellable h-vectors.
- CMness is a topological condition [Munkres 1980].

Constructibility

Definition

A simplicial complex Δ of dimension $d-1$ is constructible if either

1. Δ is a simplex; or
2. $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1 and Δ_2 are constructible of dimension $d-1$ and $\Delta_1 \cap \Delta_2$ is constructible of dimension $d-2$.

- Shellable $\implies$ constructible $\implies$ Cohen-Macaulay
- Constructibility is a nightmare to check algorithmically!

Partitionability

Definition

Δ is partitionable if its face poset (ordered by inclusion) can be written as a disjoint union

$$\bigcup_{i=1}^{k} [R_i, F_i]$$

where the F_i's are the facets.

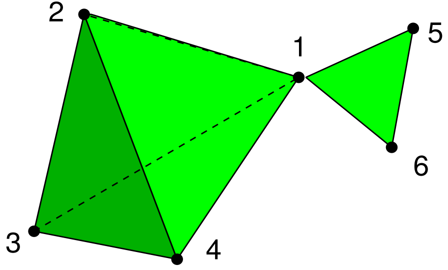
Like shellability, partitionability implies that

$$h_j = \#\{i : |R_i| = j\} \geq 0 \quad \forall j$$

but is a much weaker condition.

Partitionability

Example (Björner)



$$\Delta = [\varnothing, 156] \cup [2, 123] \cup [3, 134] \cup [4, 124] \cup [234, 234]$$

$f(\Delta) = (1,\ 3,\ 0,\ 1)$ (has a gap – not Cohen-Macaulay)

Testing Partitionability

Testing partitionability can be reduced to an integer programming problem (Sage: use `MixedIntegerLinearProgram`).

1. (Optional) Check that $h(\Delta) \geq 0$.
2. Create a MILP with
   - a binary variable $X_{r,f}$ for each pair (r, f), where f is a facet and $r \subseteq f$,
   - constraints $X_{r,f} + X_{r',f'} \leq 1$ whenever $[r, f] \cap [r', f'] \neq \varnothing$,
   - and objective function

$$\sum_{(r,f)} 2^{\dim f - \dim r} X_{r,f}$$

3. Partitionability $\iff$ optimal value = total number of faces.

Partitionability

- Shellable $\implies$ partitionable: immediate from definition
- Shellable $\implies$ Cohen-Macaulay: not too hard; easiest via constructibility
  - Converse is false (M.E. Rudin constructed a nonshellable 3-ball in 1958; more recent, smaller examples due to Grünbaum, Ziegler, others)
- Cohen-Macaulay does not imply partitionable [Duval–Goeckner–Klivans–JLM 2015$^+$], disproving a conjecture of Stanley (1980), although some special cases remain open

Relative Simplicial Complexes

Definition

A relative simplicial complex is a set family $\Omega \subseteq 2^{[n]}$ such that

$$\tau, \sigma \in \Omega,\ \tau \subseteq \rho \subseteq \sigma \quad \Longrightarrow \quad \rho \in \Omega.$$

- The face poset $F(\Omega)$ is always of the form $F(\Delta) \setminus F(\Gamma)$, where $\Gamma \subseteq \Delta$ are simplicial complexes.
- Geometrically, $[\Omega] = [\Delta]/[\Gamma]$.
- Many simplicial complex methods (link, deletion, f- and h-vectors, shellability, CMness, partitionability, ...) extend well to the relative case
- Behave better for some operations (e.g., quotients).

Simplicial Complexes in Sage

- The SimplicialComplex class contains many of the methods discussed, plus others (join, induced subcomplex, simplicial (co)homology, ...)
- Methods already tested/improved at Sage Days 74: shellability, CMness
- Methods I have written and hope to push to the server later today: partitionability (efficient), constructibility (horrible)
- What I'd like to see in the future: full support for relative complexes (I have written a class but need to figure out how to integrate it with existing Sage functionality)

Thank you!!

Merci beaucoup!!