Postroid cluster algebras and the octahedron recurrence

SAGE Days June 2015

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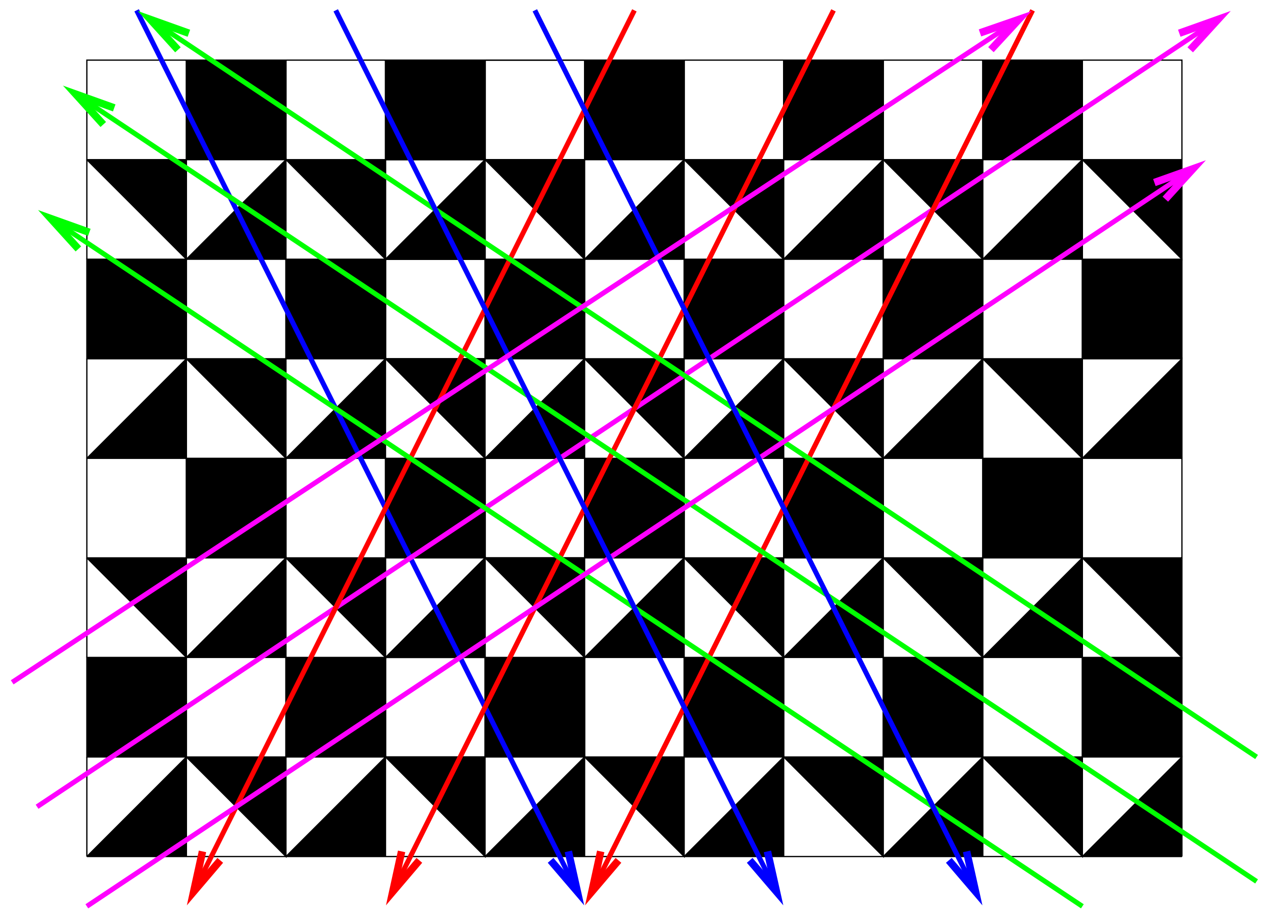


I want to tell you two stories which generalize the Grassmannian/weak separation/plabic tiling story.

- Positroid cluster algebras.
- The octahedron recurrence.

The second story is basically a special case of the first. References for the first are more available than the second.

Convention: Lower case letters (a, b, c, ...) are integers, upper case letters (A, B, C, ...) are sets of integers and calligraphic letters ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ...) are collections of sets of integers.

Recall the Grassmannian

$$G(k,n) = \mathrm{GL}_k \backslash \{M \in \mathrm{Mat}_{k\times n} : \operatorname{rank} M = k\}.$$

It will be convenient to index the columns of M by $\mathbb{Z}$, periodically mod n. We write $M_{[a,b]} = (M_a \ M_{a+1} \ \cdots \ M_b)$.

Last time we discussed a cluster structure on

$$\mathrm{GL}_k \backslash \{M : \det(M_{[a,a+k-1]}) \neq 0 \text{ for all } a\}.$$

The condition $\det(M_{[a,a+k-1]}) \neq 0$ is equivalent to $\operatorname{rank} M_{[a,b]} = \min(b-a+1,k)$.

We will be considering ***open positroid varieties***, which are subvarieties of $G(k,n)$ of the form

$$\Pi^\circ = \mathrm{GL}_k \backslash \{M : \operatorname{rank} M_{[a,b]} = r_{ab}\}$$

for some given array r_{ab}. Every point of the Grassmannian is in precisely one (open) positroid variety.

There are several different combinatorial objects which describe the combinatorics of the r_{ab}.

Running example:

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Cyclic rank matrix:

r_{ab}	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	2	2	2	3	3						
2		1	2	2	2	3	3	3					
3			1	2	2	3	3	3	3				
4				1	1	2	3	3	3	3			
5					0	1	2	2	2	3	3		
6						1	2	2	2	3	3	3	
7							1	2	2	3	3	3	3

Grassmann necklace: $I_a = \{b \bmod n : r_{ab} > r_{a(b-1)}\}$.

r_{ab}	1	2	3	4	5	6	7	8	9	10	11	12	13	
1	1	1	2	2	2	3	3							$I_1 = \{1, 3, 6\}$
2		1	2	2	2	3	3	3						$I_2 = \{2, 3, 6\}$
3			1	2	2	3	3	3	3					$I_3 = \{3, 4, 6\}$
4				1	1	2	3	3	3	3				$I_4 = \{4, 6, 7\}$
5					0	1	2	2	2	3	3			$I_5 = \{6, 7, 3\}$
6						1	2	2	2	3	3	3		$I_6 = \{6, 7, 3\}$
7							1	2	2	3	3	3	3	$I_7 = \{7, 1, 3\}$

Grassmann necklaces $(I_1, I_2, \ldots, I_n)$ are characterized by:

- $\#I_a = k$ for all a and
- $I_a \setminus \{a\} \subseteq I_{a+1}$.

Bounded affine permutations: A map $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i+n) = f(i) + n$ and $i \leq f(i) \leq i+n$ and $\frac{1}{n}\sum_{i=1}^{n}(f(i) - i) = k$.

If $I_a \neq I_{a+1}$, then $I_{a+1} = (I_a \setminus \{a\}) \cup \{f(a)\}$.

If $I_a = I_{a+1}$ then either (1) $M_a = 0$, in which case a is not in any I_b and $f(a) = a$ or (2) $M_a \notin \mathrm{span}(M_{a+1}, \ldots, M_{a+n})$, in which case a is in every I_b and $f(a) = a + n$.

$$\begin{array}{ll}
I_1 = \{1,3,6\} & f(1) = 2 \\
I_2 = \{2,3,6\} & f(2) = 4 \\
I_3 = \{3,4,6\} & f(3) = 7 \\
I_4 = \{4,6,7\} & f(4) = 10 \\
I_5 = \{6,7,3\} & f(5) = 5 \\
I_6 = \{6,7,3\} & f(6) = 8 \\
I_7 = \{7,1,3\} & f(7) = 13
\end{array}$$

To repeat myself: Our combinatorial input is, equivalently, a cyclic rank matrix r_{ab}, a Grassmann necklace I_a or a bounded affine permutation f. The geometric output is a subvariety of the Grassmannian:

$$\Pi^\circ = \mathrm{GL}_k \backslash \{M : \operatorname{rank} M_{[a,b]} = r_{ab}\}.$$

See Knutson-Lam-Speyer for a zillion geometric properties of these spaces, and their closures.

Closed positroid varieties include as special cases the Schubert varieties and the Richardson varieties. Open positroid varieties include as special cases Fulton's open matrix Schubert varieties and the preimages of double Bruhat cells in GL_n.

The closed positroid varieties are the projections of closed Richardson varieties from Flag_n.

Cluster structure on positroid varieties

Frozen variables The Plücker coordinates p_{I_a}, indexed by the Grassmann necklace.

Compatability (unchanged) I and J are weakly separated if $I \setminus J$ and $J \setminus I$ are noncrossing.

Set

$$\mathcal{M} = \left\{ J \in \binom{[n]}{k} : \#(J \cap [a,b]) \leq r_{ab} \text{ for all } [a,b] \right\}.$$

An index J is in $\mathcal{M}$ if and only if p_J is **not** identically zero on Π°.

(Some) cluster variables Plücker coordinates p_J for which $J \in \mathcal{M}$ and J is weakly separated from all I_a.

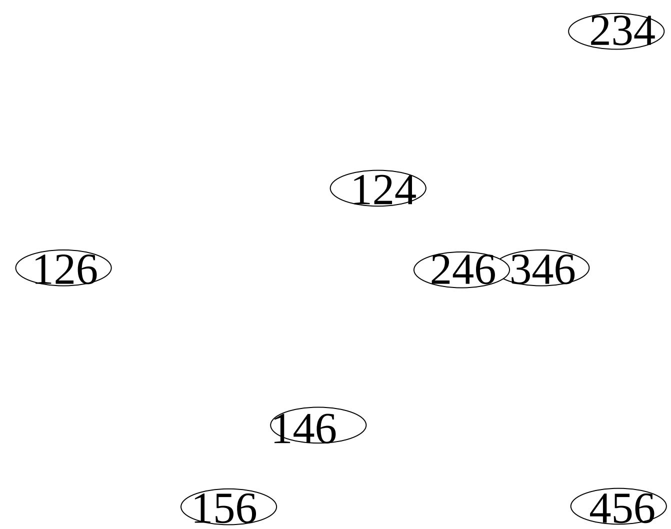
Clusters Maximal weakly separated collections.

Mutation (unchanged) The Plücker relation

$$p_{Sac}p_{Sbd} = p_{Sab}p_{Scd} + p_{Sbc}p_{Sad}.$$

Building plabic tilings

Basically, the same story. Start with a weakly separated collection $\mathcal{C}$. Take $v_1, v_2, \ldots, v_n$ at the vertices of a convex n-gon. Draw $I \in \mathcal{C}$ at position $v_I := \sum_{i \in I} v_i$.



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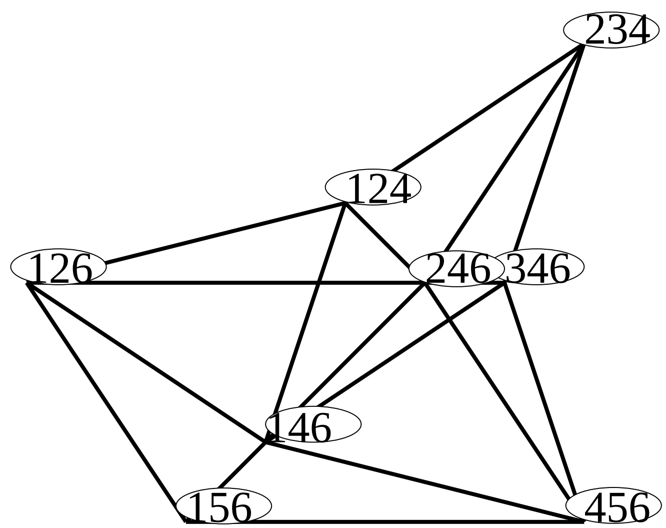

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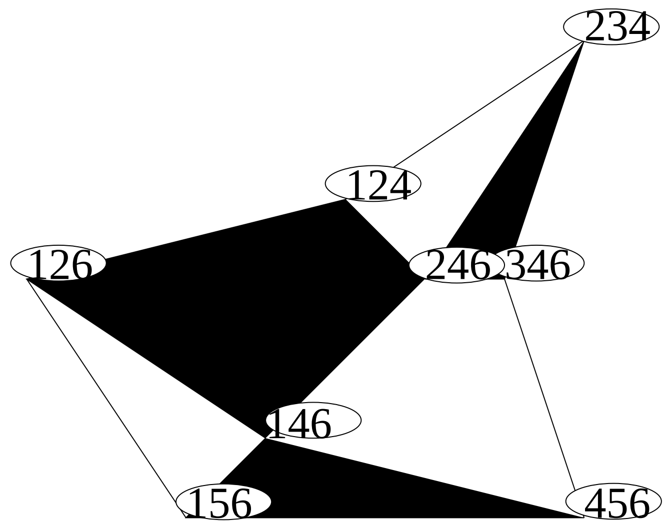

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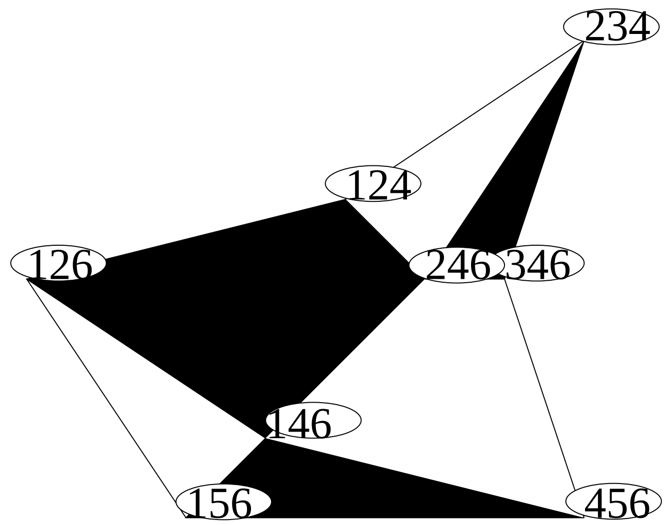
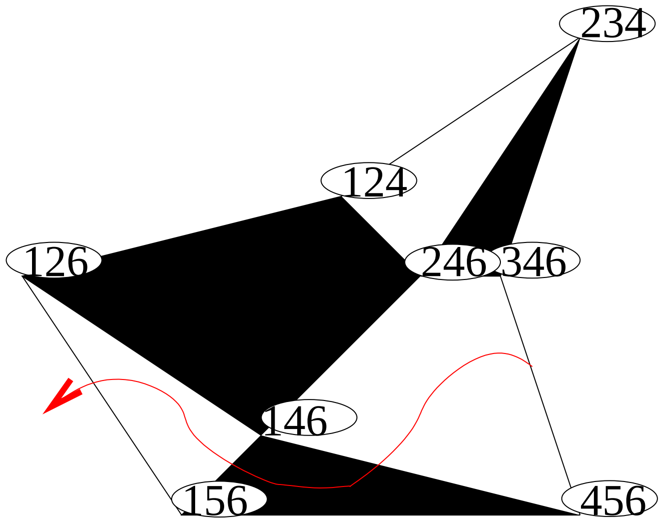


The outer boundary is still the polygonal path with vertices v_{I_1}, v_{I_2}, …, v_{I_n}. However, it need not be convex.

We draw strands the same way.



The conditions for a strand diagram to be reduced are the same as before.

The connectivity is now $a \rightsquigarrow f(a)$.

The previous construction definitely defines a cluster algebra. What is its relation to positroid varieties? Here is what we know:

- Leclerc constructs a cluster structure on Π°. Leclerc prefers not to invert frozen variables, and there are still some unanswered questions in that setting. But, if you invert the frozen variables, Leclerc's Theorem 4.5.(ii) says that his cluster algebra is the coordinate ring of Π°.

- There seem to be some subtleties regarding relating Leclerc's cluster structure to the one we discussed. I expect they will turn out to be morally the same, but the details are nontrivial.

- The cluster algebra defined from plabic combinatorics is locally acyclic (Muller-S.).

- For any maximal weakly separated collection $\mathcal{C}$, the open subset $\{x \in \Pi^\circ : p_I(x) \neq 0 \text{ for } I \in \mathcal{C}\}$ is a torus. (Muller-S., to appear).

The octahedron recurrence

I'll present this in a way that makes it appear a separate generalization of the Grassmannian story, then reveal how to encode it using positroids. I'll be a little sloppy about the underlying algebraic varieties.

Let

$$\Delta = \{(x_1, x_2, \ldots, x_n) \in [0, r]^n : \sum x_i = k\}.$$

We will be building a cluster algebra where some cluster variables $p(\vec{x})$ are indexed by $\vec{x} \in \Delta \cap \mathbb{Z}^n$.

For the Grassmannian case, take $r = 1$. Then the cluster variables are indexed by $(0, 1)$ vectors with sum k, which is equivalent to k-element subsets of $[n]$.

Convention: I write $\vec{\ }$ over points of $\mathbb{Z}^n$ or $\mathbb{R}^n$.

Compatability We define $\vec{x}$ and $\vec{y} \in \Delta$ to be weakly separated if $\{i : \vec{x}_i < \vec{y}_i\}$ and $\{j : \vec{x}_j > \vec{y}_j\}$ are weakly separated as subsets of the circularly ordered $[n]$.

Clusters Maximal weakly separated collections.

Mutation

$$
\begin{aligned}
&p(\vec{x} + \vec{e}_a + \vec{e}_c)p(\vec{x} + \vec{e}_b + \vec{e}_d) = \\
&\qquad p(\vec{x} + \vec{e}_a + \vec{e}_b)p(\vec{x} + \vec{e}_c + \vec{e}_d) + p(\vec{x} + \vec{e}_b + \vec{e}_c)p(\vec{x} + \vec{e}_a + \vec{e}_d)
\end{aligned}
$$

for $1 \leq a < b < c < d \leq n$. Here $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are the standard basis vectors in $\mathbb{Z}^n$.

Names for this recurrence

$$p(\vec{x}+\vec{e}_a+\vec{e}_c)p(\vec{x}+\vec{e}_b+\vec{e}_d) =$$
$$p(\vec{x}+\vec{e}_a+\vec{e}_b)p(\vec{x}+\vec{e}_c+\vec{e}_d)+p(\vec{x}+\vec{e}_b+\vec{e}_c)p(\vec{x}+\vec{e}_a+\vec{e}_d)$$

The 6 points $\vec{x}+\vec{e}_a+\vec{e}_b$, $\vec{x}+\vec{e}_a+\vec{e}_c$, $\vec{x}+\vec{e}_a+\vec{e}_d$, $\vec{x}+\vec{e}_b+\vec{e}_c$, $\vec{x}+\vec{e}_b+\vec{e}_c$, $\vec{x}+\vec{e}_b+\vec{e}_d$ and $\vec{x}+\vec{e}_c+\vec{e}_d$ lie at the vertices of an octahedron, hence the term ***octahedron recurrence***.

When $n=4$, a change of basis turns this into a recursion in $\mathbb{Z}^3$:

$$p(\vec{y}+\vec{f}_1)p(\vec{y}-\vec{f}_1) = p(\vec{y}+\vec{f}_2)p(\vec{y}-\vec{f}_2)+p(\vec{y}+\vec{f}_3)p(\vec{y}-\vec{f}_3).$$

Hirota wrote it in this form, and it is often studied under the names ***Hirota's bilinear difference equation*** or ***Hirota-Miwa equation***. Saito seems to have been first to notice the n-dimensional version.

Plabic tilings

Choose a linear map $\pi : \mathbb{R}^n \to \mathbb{R}^2$ such that $\pi(\vec{e}_1)$, $\pi(\vec{e}_2)$, $\ldots$, $\pi(\vec{e}_n)$ are the vertices of a convex n-gon.

Draw $\vec{x}$ in location $\pi(\vec{x})$.

Define $\vec{x}$ and $\vec{y}$ to be ***adjacent*** if $\vec{x} - \vec{y}$ is of the form $e_a - e_b$.

Again, there are two kinds of cliques in the adjacency graph:

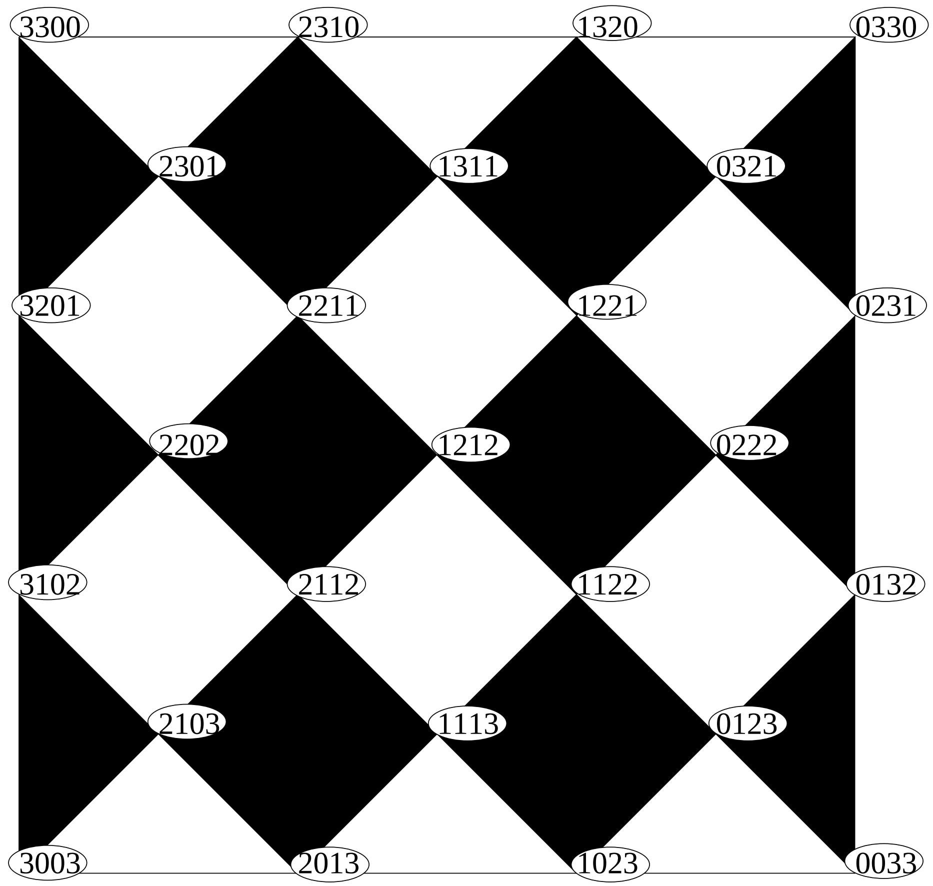
$$\{\vec{y} + e_{a_1}, \vec{y} + e_{a_2}, \ldots, \vec{y} + e_{a_r}\}$$

$$\{\vec{z} + e_{b_1}, \vec{z} + e_{b_2}, \ldots, \vec{z} + e_{b_s}\}$$

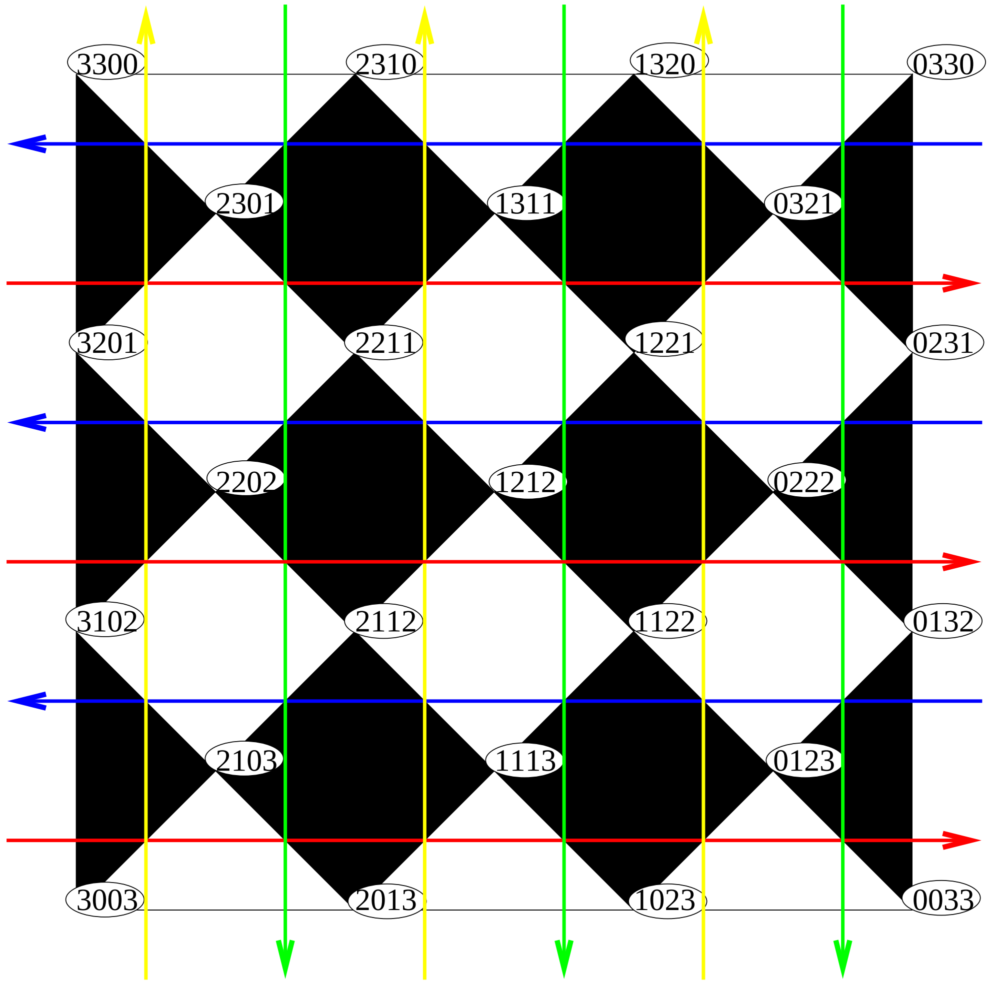
Color the cliques black and white to obtain a tiling of $\pi(\Delta)$.

Triangulating the tilings gives sections of $\pi : \Delta \to \pi(\Delta)$.

Disclaimer/Apology I don't believe this material has been carefully written down anywhere.



$$\Delta = \{(x_1, x_2, x_3, x_4) \in [0,3]^4 : x_1 + x_2 + x_3 + x_4 = 6\}.$$



Strands separate $\vec{x}_j \geq c$ and $\vec{x}_j < c$. In the example, yellow, red, green and blue strands correspond to $j = 1$, 2, 3 and 4.

One often wants to think “in the limit as $r \to \infty$, and $k \approx rn/2$”, in which case one obtains tilings of all of $\mathbb{R}^2$. This is some sort of infinite rank cluster algebra.

One can then return to finite rank by imposing periodicity modulo some rank $n-2$ lattice. This gives rise to plabic tilings/alternating strand diagrams on a torus. See Goncharov-Kenyon and Eager-Franco for more.

Encoding the octahedron recurrence in a positroid

We were working with lattice points $(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)$ in $\{0, 1, 2, \ldots, r\}^n$.

We can encode these by lattice points in $(0, 1)^{rn}$: Simply write

$$(\overbrace{0, 0, \ldots, 0}^{r-\vec{x}_j}, \overbrace{1, 1, \ldots, 1}^{\vec{x}_j})$$

in the $rj + 1$ through $r(j + 1)$ positions.

This gives a bijection between weakly separated collections for the multidimensional octahedron recurrence, and weakly separated collections for a certain positroid variety Π°.

Π° should be something like the space of configurations of k partial flags of dimensions $(1, 2, \ldots, r)$ in n-space.