Combinatorial underpinnings of Grassmannian cluster algebras

SAGE Days June 2015

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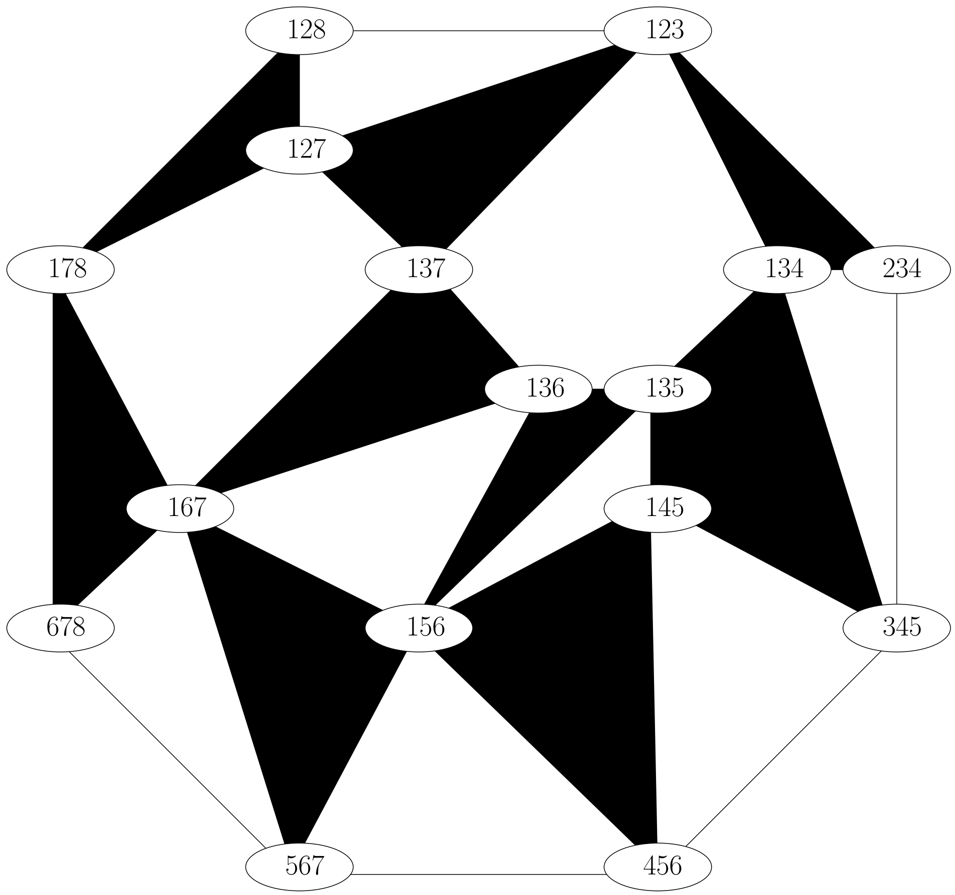


Some of the most beautiful and important cluster algebras are the cluster structures on the Grassmannian and related spaces. By related spaces I mean positroid varieties, flag manifolds, (type A) double Bruhat cells, configuration spaces of flags, the space of solutions to the (multidimensional) octahedron recurrence (aka discrete Toda lattice) .

There are numerous combinatorial devices which have been invented to describe these cluster structures. They are elegant but unwieldy. I would love it if SAGE had data structures to natively work with them.

In this talk, I'll talk about the cluster structure on the Grassmannian itself. If people are interested, I would love to explain how to get to all the other spaces.

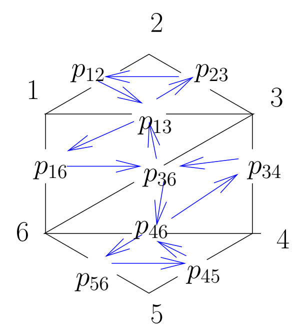
$G(2,n)$ – we want every cluster algebra to be this nice

Cluster variables: p_{ab}, for $1 \leq a < b \leq n$.

Frozen variables: p_{12}, p_{23}, $\ldots$, $p_{(n-1)n}$, p_{1n} .

Compatibility: p_{ab} and p_{cd} may occur in a common cluster as long as (a,b) and (c,d) are non crossing.

Mutation: $p_{ac}p_{bd} = p_{ab}p_{cd} + p_{ad}p_{bc}$ for $1 \leq a < b < c < d \leq n$.



Clusters: Triangulations of the n-gon.

B-matrix/Quiver: Connect neighboring chords of the triangulation.

Underlying space: $\operatorname{Proj} \mathbb{C}[p_{ab}]$ is the Grassmannian

$$G(2,n) = \mathrm{GL}_2 \backslash \{M \in \mathrm{Mat}_{2\times n} : \mathrm{rank}(M) = 2\}.$$

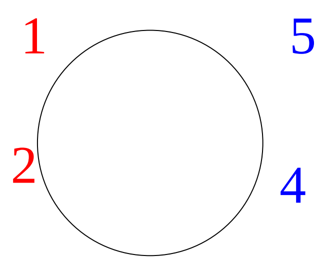
We have $p_{ab} = \det M_{ab}$. So inverting the frozen variables imposes $\det(M_{12})$, $\det(M_{23})$, $\ldots$, $\det(M_{(n-1)n})$, $\det(M_{1n}) \neq 0$.

Seeking an analogous story for $G(k,n)$

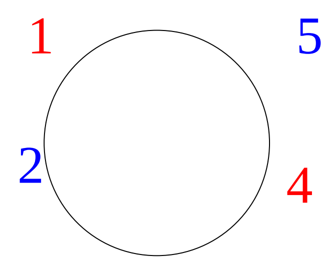
(Some) cluster variables: p_I with $I = \{i_1, i_2, \ldots, i_k\} \subset [n]$.

Frozen variables: $p_{12\cdots k},\ p_{23\cdots k(k+1)},\ \ldots,\ p_{(n-k+1)\cdots(n-1)n},$
$p_{(n-k+2)\cdots n1},\ \ldots,\ p_{n12\cdots(k-1)}.$

Compatibility Given I and $J \in \binom{[n]}{k}$, we say I and J are ***weakly separated*** if we can draw a chord separating $I \setminus J$ and $J \setminus I$.



123 and 345 are weakly separated.



134 and 235 are **not** weakly separated.

Seeking an analogous story for $G(k,n)$ continued

Mutation Our most basic mutation relation will be the three term Plücker relation:

$$p_{Sac}p_{Sbd} = p_{Sab}p_{Scd} + p_{Sbc}p_{Sad} \qquad 1 \leq a < b < c < d \leq n.$$

Underlying space: $\operatorname{Proj} \mathbb{C}[p_I]$ is the Grassmannian

$$G(k,n) = \mathrm{GL}_k \backslash \{M \in \mathrm{Mat}_{k\times n} : \mathrm{rank}(M) = k\}.$$

The cluster variable $p_{i_1 i_2 \cdots i_k}$ will be the Plücker variable $\det(M_{i_1} \cdots M_{i_k})$. Inverting the frozen variables imposes that $\det(M_i \ M_{i+1} \ \cdots \ M_{i+k-1}) \neq 0$.

So far we have described …

Some cluster variables: Plucker coordinates $p_{i_1 i_2 \cdots i_k}$.

The rest of the cluster variables? I won't talk about this, but see Fomin and Pylyavskyy for some beautiful conjectures when $k = 3$.

Frozen variables: Cyclically consecutive minors $p_{i(i+1)\cdots(i+k-1)}$.

Compatibility: Weak separation, meaning that $I \setminus J$ and $J \setminus I$ are in separate arcs.

Mutation: Plücker relation $p_{Sac}p_{Sbd} = p_{Sab}p_{Scd} + p_{Sbc}p_{Sad}$.

Combinatorial model for clusters?

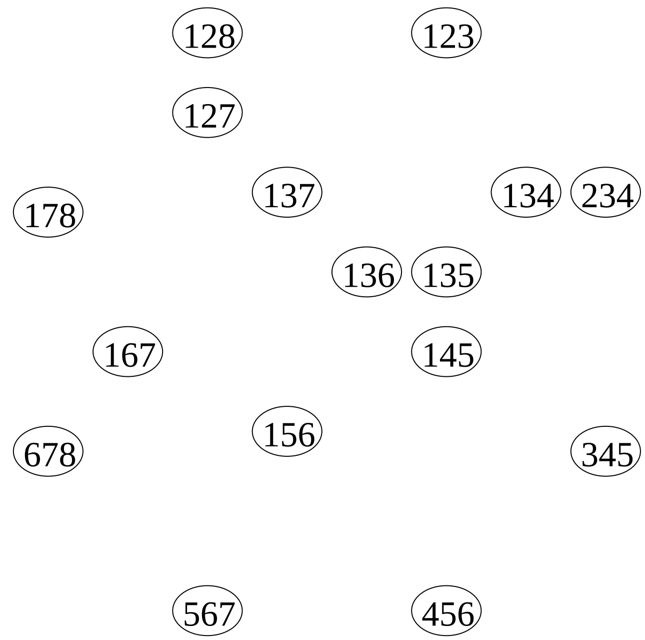
B-matrix/Quiver?

Underlying space: The Grassmannian $G(k,n)$, or the open locus $\det(M_i M_{i+1} \cdots M_{i+k-1}) \neq 0$ inside it.

History: Weak separation was first studied by LeClerc and Zelevinsky, who showed that it is the condition for quantum minors to quasi-commute on the flag manifold. See also Scott.

Notational note: We use lower case letters for integers: a, b, c, d, k, n and capital letters for sets of integers: I, J, S. We will soon use calligraphic letters $\mathcal{C}$, $\mathcal{I}$ for collections of sets of integers.

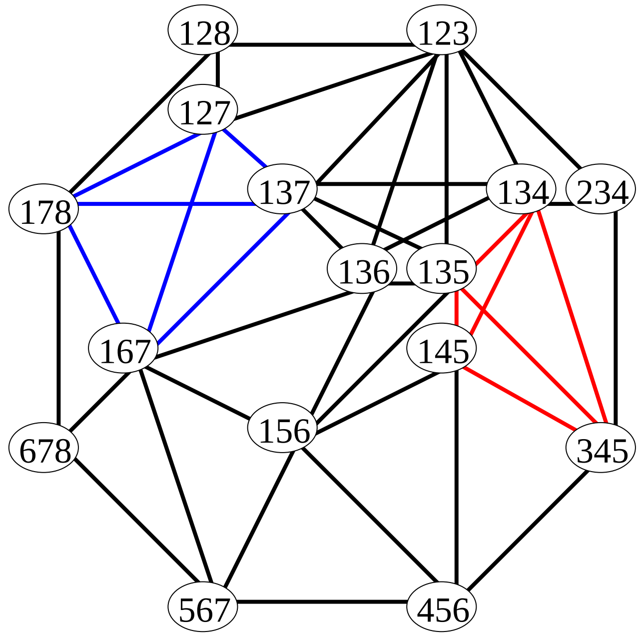
A picture analogous to triangulations



Let $\mathcal{C}$ be a weakly separated collection in $\binom{[n]}{k}$. (Soon we will impose that it is maximal, but not yet.) We want to assign some sort of two dimensional diagram to it.

Choose v_1, v_2, $\ldots$, v_n in $\mathbb{R}^2$ at the vertices of a convex n-gon. For $I \in \binom{[n]}{k}$, set $v_I = \sum_{i \in I} v_i$. We draw $I \in \mathcal{C}$ in the location v_I.

A picture analogous to triangulations continued



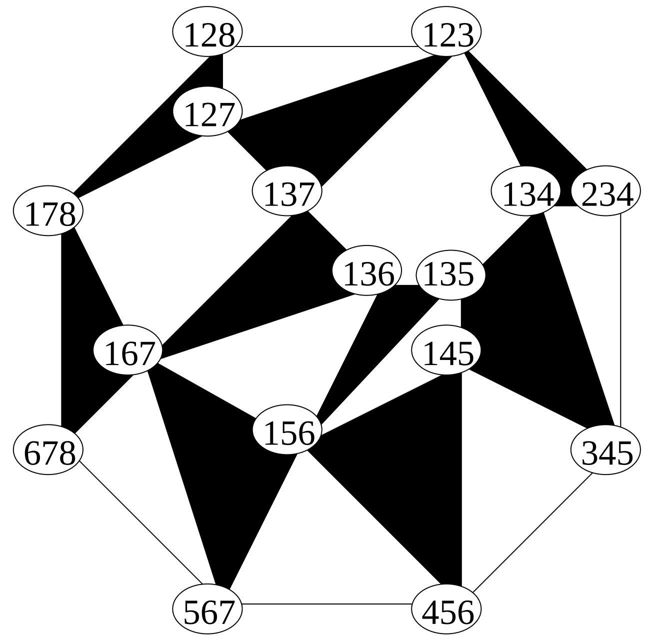
Define I and J in $\mathcal{C}$ to be adjacent if $\#(I \setminus J) = \#(J \setminus I) = 1$.

In the adjacency graph, there are two sorts of cliques:

$$\{K \cup a_1, K \cup a_2, \ldots, K \cup a_r\}$$

$$\{L \setminus b_1, L \setminus b_2, \ldots, L \setminus b_s\}$$

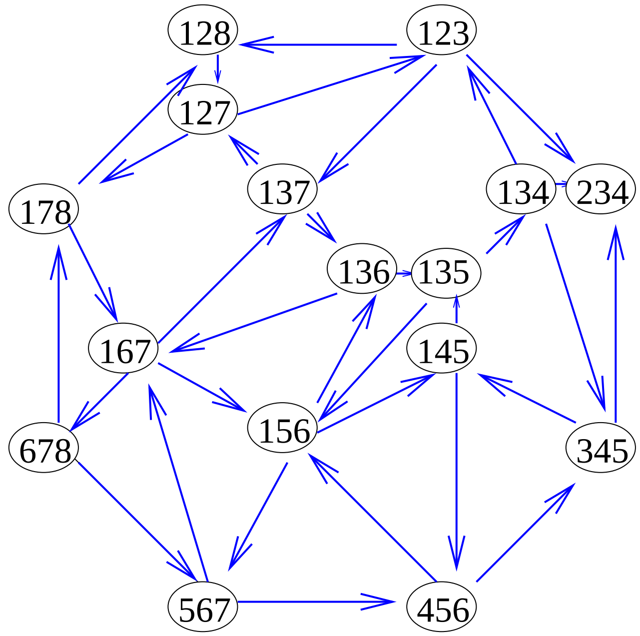
A picture analogous to triangulations continued



The cliques form convex polygons which we color white and black. We call this a ***plabic tiling***.

Theorem (Oh-Postnikov-Speyer) The plabic tiling is a 2-dimensional CW-complex embedded in $\mathbb{R}^2$. It is contained inside the convex n-gon $\mathrm{Hull}(v_I)_{I \text{ frozen}}$. The plabic tiling fills the interior of $\mathrm{Hull}(v_I)_{I \text{ frozen}}$ if and only if $\mathcal{C}$ is maximal.

A picture analogous to triangulations continued



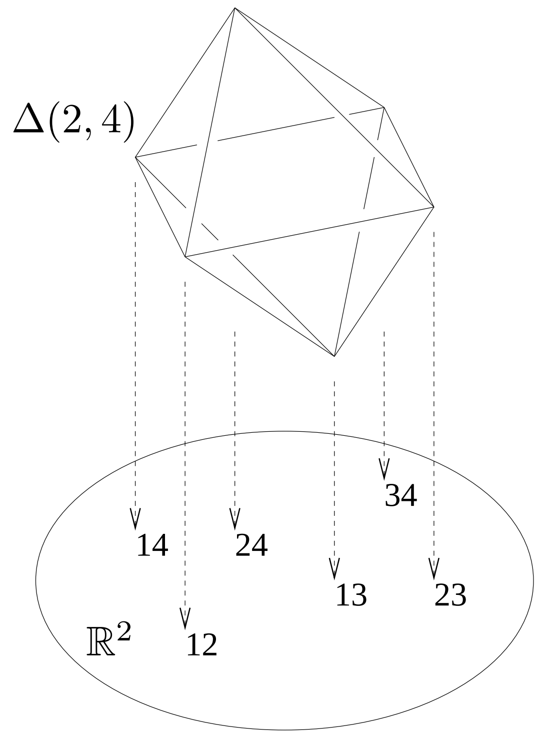
Theorem (Oh-Postnikov-Speyer) The maximal weakly separated collections are precisely those clusters of $G(k,n)$ all of whose elements are Plücker variables. We obtain the quiver of a cluster by orienting the polygons cyclically, according to their color.

Useful algorithmic task Complete a weakly separated collection to a maximal one.

Useful algorithmic task Draw these pictures.

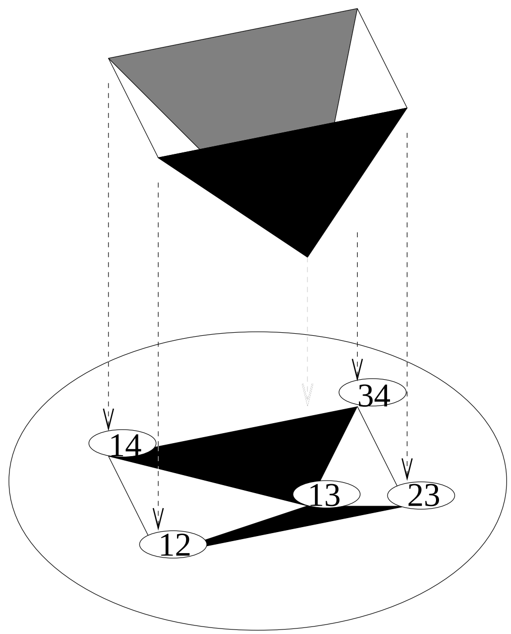
Conjecture Every tiling of $\mathrm{Hull}(v_i + v_{i+1} + \cdots + v_{i+k-1})$ by polygons of the form $x + \mathrm{Hull}(v_{a_1}, v_{a_2}, \ldots, v_{a_r})$ and $y - \mathrm{Hull}(v_{b_1}, v_{b_2}, \ldots, v_{b_s})$ is a plabic tiling.

Connections to polyhedral geometry



Let $\Delta(k,n)$ be the ***hypersimplex*** $\mathrm{Hull}(e_{i_1}+\cdots+e_{i_k}) \subset \mathbb{R}^n$. Let the linear map $\pi : e_i \to v_i$ from $\mathbb{R}^n \to \mathbb{R}^2$ take e_i to v_i, so π takes the vertices of $\Delta(k,n)$ to the v_I.

Connections to polyhedral geometry continued

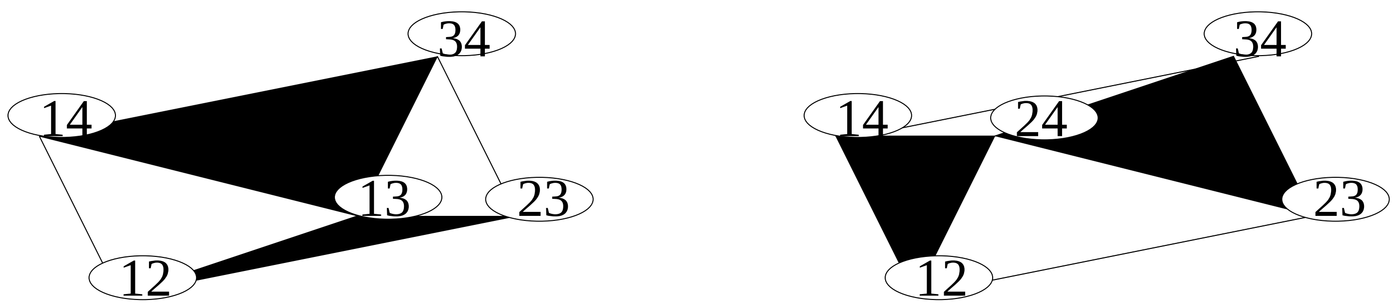


Using a triangulation of our tiling, we can obtain a piece-wise linear section $\sigma : \pi(\Delta(k,n)) \to \Delta(k,n)$, landing in the 2-skeleton of $\Delta(k,n)$. White triangles land in faces of the form $\mathrm{Hull}(v+e_i, v+e_j, v+e_k)$; black triangles land in faces of the form $\mathrm{Hull}(v-e_i, v-e_j, v-e_k)$.

Changing the triangulation of a plabic tiling moves our section σ across a tetrahedral face of $\Delta(k,n)$.

Moving across on octahedral face is ***mutation***

$$\{Sab, Sbc, Scd, Sad, \boldsymbol{Sac}\} \iff \{Sab, Sbc, Scd, Sad, \boldsymbol{Sbd}\}.$$



Theorems about mutation

Theorem (Morally Postnikov) Any two maximal weakly separated collections can be connected by a sequence of mutations.

Theorem (Danilov-Karzanov-Koshevoy, Oh-Speyer) Any two maximal weakly separated collections $\mathcal{A}$ and $\mathcal{B}$ can be connected by a sequence of mutations where all intermediate sets contains $\mathcal{A} \cap \mathcal{B}$.

Write $d(\mathcal{A}, \mathcal{B})$ for $\#(\mathcal{A} \setminus \mathcal{B}) = \#(\mathcal{B} \setminus \mathcal{A})$. The Oh-Speyer proof shows that, if $d(\mathcal{A}, \mathcal{C}) = r$, then either

- We can find $\mathcal{B}$ with $\mathcal{B} \supset \mathcal{A} \cap \mathcal{C}$ such that $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B}, \mathcal{C}) = r - 1$ or else
- We can find $\mathcal{B}_1$ and $\mathcal{B}_2$ with $\mathcal{B}_1, \mathcal{B}_2 \supset \mathcal{A} \cap \mathcal{C}$ such that $d(\mathcal{A}, \mathcal{B}_1) = d(\mathcal{B}_2, \mathcal{C}) = r - 1$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ differ by a mutation.

This shows that it takes at most $2^{d(\mathcal{A}, \mathcal{B})} - 1$ mutations to connect $\mathcal{A}$ to $\mathcal{B}$.

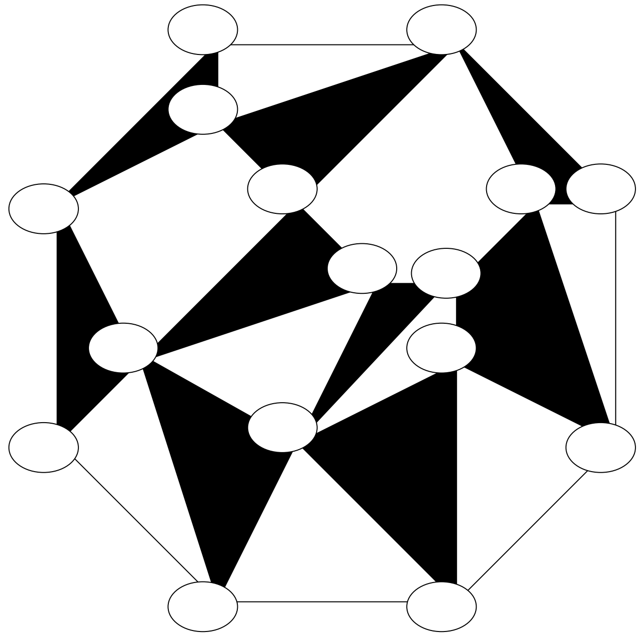
Open Problem What is the true bound for the number of mutations needed?

Useful algorithmic task Given two maximal weakly separated collections, find a chain of mutations linking them.

Open Problem Consider the simplicial complex whose vertices are $\binom{[n]}{k}$ and whose faces are the weakly separated collections. We've shown that this complex is pure dimensional and that it (and the link of every face within it) is connected in codimension 1. What else can be said about its topology? See Hess and Hirsch for some preliminary results; much left to do.

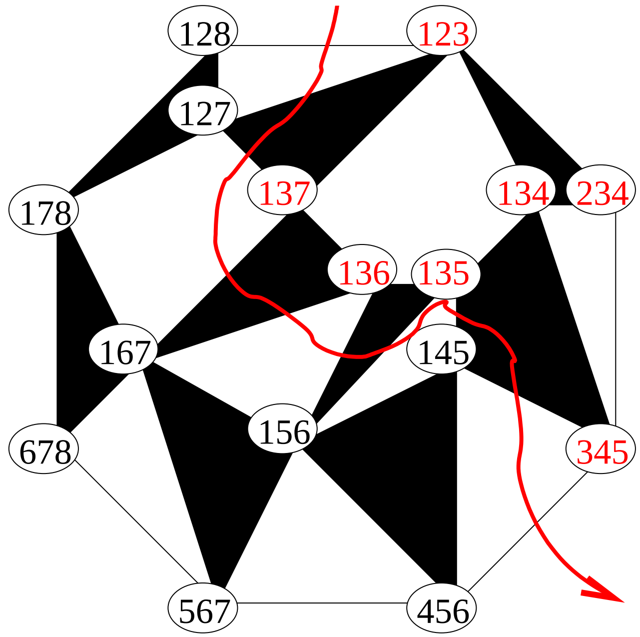
Open problem Let Δ be a polytope in $\mathbb{R}^n$ and $\pi : \Delta \to \mathbb{R}^d$ a linear map. Billera, Kapranov and Sturmfels define a poset of ***locally coherent strings***, whose minimal elements are the piecewise linear sections $\pi(\Delta) \to \Delta$ with image landing in the d-skeleton of Δ. They posed the ***generalized Baues conjecture***, that this poset is homotopy equivalent to a sphere, but this was disproved by Rambau and Ziegler. Does it hold for the case of π and $\Delta(k, n)$?

Alternating strand diagrams



Suppose that we are given the topological picture of $\Sigma(\mathcal{C})$ but not the labeling of the vertices by $\binom{[n]}{k}$. How can we recover it?

Alternating strand diagrams continued



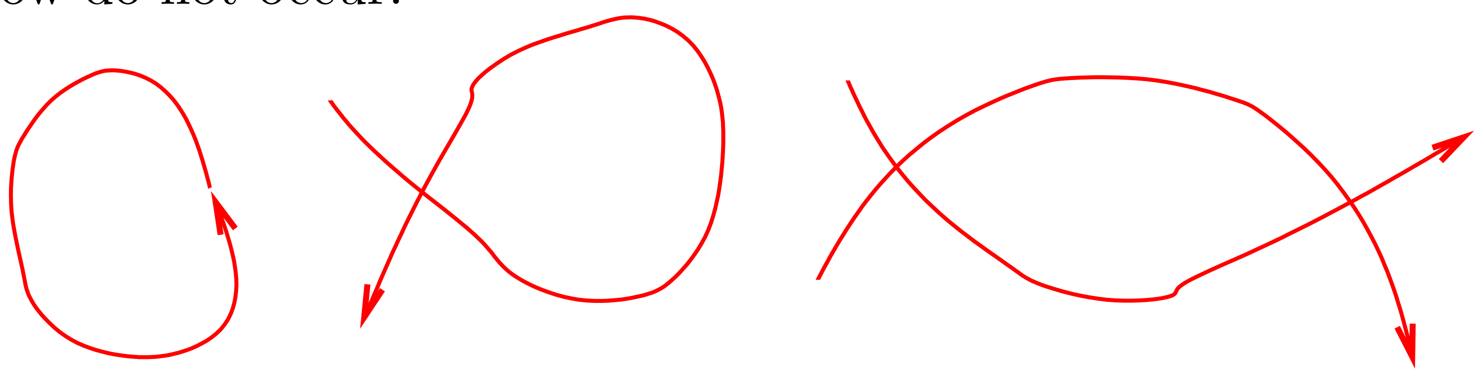
We draw "strands" cutting across the corners of the tiles, moving clockwise on white tiles and counterclockwise on black tiles. The vertex labels to the left of the i-th strand contain i, the vertex labels to the right do not.

Alternating strand diagrams continued

We can apply this recipe to an arbitrary bicolored tiling T of a disc by polygons.

Theorem (Postnikov, Oh-Postnikov-Speyer) The tiling T is of the form $\Sigma(\mathcal{C})$ for some $\mathcal{C}$ if and only if

- The strands are ***reduced***, meaning that the configurations below do not occur:



- The strands have the correct ***connectivity***, from $i-k$ to i.

If I had more time ...

I can keep the local aspects of our models and discard the global conditions. For example, I can fit together polygons of the $x + \mathrm{Hull}(v_{a_1}, v_{a_2}, \ldots, v_{a_i})$ and $x - \mathrm{Hull}(v_{b_1}, v_{b_2}, \ldots, v_{b_j})$ but tile other planar regions. This corresponds to reduced alternating strand diagrams with connectivity other than $i - k \rightsquigarrow i$.

This leads us to positroids and positroid varieties, which include a wide variety of interesting cluster algebras.

Thank you.

Historical notes

Scott originally worked with reduced alternating strand diagrams (which she called Postnikov diagrams) without drawing the underlying tiling.

Postnikov works with the duals of plabic tilings, which he calls plabic (planar bicolored) graphs.

Scott proved that reduced alternating strand diagrams correspond to cluster structures on $G(k, n)$. Postnikov proved connectivity under mutations for reduced plabic graphs.

Oh, Postnikov and Speyer proved that maximal weakly separated collections correspond to reduced plabic graphs.

Danilov, Kharzanov and Koshevoy introduced a related formalism of "generalized tilings" to model weakly separated sets. They were the first to establish that every maximal weakly separated collection has cardinality $k(n-k)+1$.

References

Billera, Kapranov and Sturmfels "Cellular strings on polytopes", Proc. AMS 1994 **122** pp. 549–555

Danilov, Karzanov and Koshevoy, "On maximal weakly separated set-systems", JACO 2010 **32** 497–531, `0909.1423`

Danilov, Karzanov and Koshevoy, "The purity of set-systems associated to Grasssmann necklaces", FPSAC 2014, `1312.3121`

Fomin and Pylyavskyy, "Tensor diagrams and cluster algebras", `1210.1888`

Hess and Hirsch "On the topology of weakly and strongly separated set complexes", Topology and its Applications, Volume 160, Issue 2, 1 February 2013, Pages 328 – 336, `1110.0880`

Leclerc and Zelevisky "Quasicommuting families of quantum Plücker coordinates", Kirillov's seminar on representation theory, 85–108, Amer. Math. Soc. Transl. Ser. 2, 181 1998.

Oh, Postnikov and Speyer "Weak Separation and Plabic Graphs", Proc. LMS to appear, `1109.4434`

Oh, Speyer "Links in the complex of weakly separated collections" `1405.5191`

Postnikov "Total Positivity, Grassmannians and Networks", `http://www-math.mit.edu/~apost/papers/tpgrass.pdf`

Rambau and Ziegler "Projections of polytopes and the generalized Baues conjecture", Discrete and Computational Geometry March 1996, Volume 16, Issue 3, pp 215-237

Scott "Quasi-commuting families of quantum minors", J. Algebra 290 2005, no. 1, 204–220 `math/0008100`

Scott "Grassmannians and Cluster Algebras", Proc. LMS 2006 Volume 92, Issue 2, 345–380. `math/0311148`