NOTES ON SCATTERING DIAGRAMS

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1. Elementary transformations

1.1. **Formal elementary transformations.** Let B be an $r \times r$ skew-symmetric integral matrix. Define the ring

$$\widehat{\mathcal{F}}(\mathsf{B}) := \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_r^{\pm 1}][[y_1, y_2, ..., y_r]]$$

that is, formal power series in the variables $y_1, y_2, ..., y_r$ with coefficients in the ring of Laurent polynomials in $x_1, x_2, ..., x_r$. We use multinomial notation for the x and y variables; that is,

$$\forall m = (m_1, m_2, ..., m_r) \in \mathbb{Z}^r, \quad x^m := \prod_{i=1}^{r} x_i^{m_i}, \quad \forall n = (n_1, n_2, ..., n_r) \in \mathbb{N}^r, \quad y^n := \prod_{i=1}^{r} y_i^{n_r}$$

For $n \in \mathbb{N}^r$, let $\gcd(n)$ denote the greatest common divisor of the coordinates of n. For non-zero $n \in \mathbb{N}^r$, define the **formal elementary transformation** $E_{n,\mathsf{B}} : \widehat{\mathcal{F}} \longrightarrow \widehat{\mathcal{F}}$ by[1]

$$E_{n,\mathsf{B}}(x^m) = (1 + x^{\mathsf{B}n} y^n)^{\frac{n \cdot m}{\gcd(n)}} x^m, \quad E_{n,\mathsf{B}}(y^{n'}) = y^{n'}$$

Here, $n \cdot m = \sum_i n_i m_i$ denotes the Euclidean inner product. We will write E_n for $E_{n,\mathsf{B}}$ when the matrix B is clear.

More generally, for $d \in \mathbb{Q}$, we may define a fractional power $E_{n,\mathsf{B}}^d : \mathbb{Q} \otimes \widehat{\mathcal{F}} \longrightarrow \mathbb{Q} \otimes \widehat{\mathcal{F}}$ by

$$E_{n,\mathsf{B}}^d(x^m) = (1 + x^{\mathsf{B}n} y^n)^{d\frac{n \cdot m}{\gcd(n)}} x^m, \quad E_{n,\mathsf{B}}^d(y^{n'}) = y^{n'}$$

Here, we explicitly use the formal series expansion

$$\begin{aligned} E_{\mathsf{B},n}^d(x^m) &= \left(\sum_{i=0}^{\infty} \binom{d\frac{n \cdot m}{\gcd(n)}}{i} x^{i\mathsf{B}n} y^{in} \right) x^m \\ &= \left(\sum_{i=0}^{\infty} \frac{(d\frac{n \cdot m}{\gcd(n)})(d\frac{n \cdot m}{\gcd(n)} - 1) \cdots (d\frac{n \cdot m}{\gcd(n)} - i + 1)}{i!} x^{i\mathsf{B}n} y^{in} \right) x^m \end{aligned}$$

The following proposition is immediate.

Proposition 1.1.1. *For all $d, d' \in \mathbb{Q}$, $E_n^d E_n^{d'} = E_n^{d+d'}$.*

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[1]Note that $\frac{n \cdot m}{\gcd(n)}$ is an integer but may be negative; however, $(1 + x^{\mathsf{B}n} y^n)^{-1}$ exists in $\widehat{\mathcal{F}}$.

In particular, since E_n^0 is the identity, each E_n^d is an automorphism with inverse E_n^{-d}.

The following lemma is useful for determining when the $\gcd(n)$ in E_n is trivial.

Lemma 1.1.2. *If $n \cdot \mathsf{B}n' = 1$, then $\gcd(an + bn') = \gcd(a, b)$ for all $a, b \in \mathbb{Z}$.*

In particular, if $n \cdot \mathsf{B}n' = 1$, then $\gcd(n) = \gcd(n') = 1$.

The following proposition records the two most fundamental commutation relations.

Proposition 1.1.3. *Let $n, n' \in \mathbb{N}^r$. (a) If $n \cdot \mathsf{B}n' = 0$, then*

$$E_n E_{n'} = E_{n'} E_n$$

(b) If $n \cdot \mathsf{B}n' = 1$, then

$$E_n E_{n'} = E_{n'} E_{n+n'} E_n$$

Proof. Since B is skew-symmetric, $n \cdot \mathsf{B}n = n' \cdot \mathsf{B}n' = 0$ and $n' \cdot \mathsf{B}n = -n \cdot \mathsf{B}n'$.

Case (a).

$$\begin{aligned} E_n E_{n'}(x^m) &= E_n\left((1 + x^{\mathsf{B}n'} y^{n'})^{\frac{n' \cdot m}{\gcd(n')}} x^m\right) \\ &= (1 + x^{\mathsf{B}n'} y^{n'})^{\frac{n' \cdot m}{\gcd(n')}} (1 + x^{\mathsf{B}n} y^n)^{\frac{n \cdot m}{\gcd(n)}} x^m \end{aligned}$$

By symmetry, $E_{n'} E_n(x^m)$ is the same.

Case (b). By the lemma, $\gcd(n) = \gcd(n') = \gcd(n + n') = 1$.

$$\begin{aligned} E_n E_{n'}(x^m) &= E_n\left((1 + x^{\mathsf{B}n'} y^{n'})^{n' \cdot m} x^m\right) \\ &= (1 + (1 + x^{\mathsf{B}n} y^n)^{n \cdot Bn'} x^{\mathsf{B}n'} y^{n'})^{n' \cdot m} (1 + x^{\mathsf{B}n} y^n)^{n \cdot m} x^m \\ &= (1 + x^{\mathsf{B}n'} y^{n'} + x^{\mathsf{B}(n+n')} y^{n+n'})^{n' \cdot m} (1 + x^{\mathsf{B}n} y^n)^{n \cdot m} x^m \end{aligned}$$

For comparison, $E_{n'} E_{n+n'} E_n(x^m)$ is equal to

$$\begin{aligned} &= E_{n'} E_{n+n'}\left((1 + x^{\mathsf{B}n} y^n)^{n \cdot m} x^m\right) \\ &= E_{n'}\left((1 + (1 + x^{\mathsf{B}(n+n')} y^{n+n'})^{-1} x^{\mathsf{B}n} y^n)^{n \cdot m} (1 + x^{\mathsf{B}(n+n')} y^{n+n'})^{n \cdot m + n' \cdot m} x^m\right) \\ &= E_{n'}\left((1 + x^{\mathsf{B}(n+n')} y^{n+n'} + x^{\mathsf{B}n} y^n)^{n \cdot m} (1 + x^{\mathsf{B}(n+n')} y^{n+n'})^{n' \cdot m} x^m\right) \\ &= E_{n'}\left((1 + (1 + x^{\mathsf{B}n'} y^{n'}) x^{\mathsf{B}n} y^n)^{n \cdot m} (1 + x^{\mathsf{B}(n+n')} y^{n+n'})^{n' \cdot m} x^m\right) \\ &= (1 + x^{\mathsf{B}n} y^n)^{n \cdot m} (1 + (1 + x^{\mathsf{B}n'} y^{n'})^{-1} x^{\mathsf{B}(n+n')} y^{n+n'})^{n' \cdot m} (1 + x^{\mathsf{B}n'} y^{n'})^{n' \cdot m} x^m \\ &= (1 + x^{\mathsf{B}n} y^n)^{n \cdot m} (1 + x^{\mathsf{B}n'} y^{n'} + x^{\mathsf{B}(n+n')} y^{n+n'})^{n' \cdot m} x^m \end{aligned}$$

Hence, they coincide. ∎

Remark 1.1.4. In general, the commutation relations between E_n and $E_{n'}$ will depend on $n \cdot \mathsf{B}n'$, $\gcd(n)$ and $\gcd(n')$ (see Theorem **??**).

Exercise 1.1. Let $n, n' \in \mathbb{N}^r$ such that $n \cdot \mathsf{B}n' = 1$. Show that

(1) $E_n^2 E_{n'} = E_{n'} E_{n+n'}^2 E_{2n+n'} E_n^2$.
(2) $E_n^3 E_{n'} = E_{n'} E_{n+n'}^3 E_{3n+2n'} E_{2n+n'}^3 E_{3n+n'} E_n^3$.

Hint: This can be done by repeatedly using Proposition 1.1.3.

Exercise 1.2. Show that, if $\mathsf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then

$$E_{(1,0)}^b E_{(0,1)}^c (x^{(m_1,m_2)}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{cm_2}{j} \binom{b(m_1+j)}{i} x^{(m_1+j,m_2-i)} y^{(i,j)}$$

1.2. **Reductions of formal elementary transformations.** As is often the case with formal rings, it is useful to be able to explore the behavior of $\widehat{\mathcal{F}}$ and E_n^d to various bounded orders.

Let I be a monomial ideal in $\mathbb{Q}[[y_1, y_2, ..., y_r]]$. Every transformation E_n^d descends to an automorphism of the quotient $\widehat{\mathcal{F}}/I$, and this automorphism is trivial if and only if $y^n \in I$.

Naturally, it is easier to find commutation relations on $\widehat{\mathcal{F}}/I$ than on $\widehat{\mathcal{F}}$.

Lemma 1.2.1. *Let $n, n' \in \mathbb{N}^r$. (a) If $y^{n+n'} \in I$, then*

$$E_n E_{n'} \equiv E_{n'} E_n \mod I$$

(b) If $y^{2n+n'}, y^{n+2n'} \in I$, then

$$E_n E_{n'} \equiv E_{n'} E_{n+n'}^{\lambda} E_n \mod I, \text{ where } \lambda := \frac{(n \cdot \mathsf{B}n') \gcd(n+n')}{\gcd(n)\gcd(n')}$$

Proof. Case (b). We evaluate $E_n E_{n'}(x^m)$.

$$\begin{aligned}
&= E_n \left((1 + x^{\mathsf{B}n'} y^{n'})^{\frac{n' \cdot m}{\gcd(n')}} x^m \right) \\
&= (1 + (1 + x^{\mathsf{B}n} y^n)^{\frac{n \cdot \mathsf{B}n'}{\gcd(n)}} x^{\mathsf{B}n'} y^{n'})^{\frac{n' \cdot m}{\gcd(n')}} (1 + x^{\mathsf{B}n} y^n)^{\frac{n \cdot m}{\gcd(n)}} x^m \\
&= \left(1 + x^{\mathsf{B}n'} y^{n'} + \frac{n \cdot \mathsf{B}n'}{\gcd(n)} x^{\mathsf{B}(n+n')} y^{n+n'} \right)^{\frac{n' \cdot m}{\gcd(n')}} (1 + x^{\mathsf{B}n} y^n)^{\frac{n \cdot m}{\gcd(n)}} x^m \\
&= \left((1 + x^{\mathsf{B}n'} y^{n'})^{\frac{n' \cdot m}{\gcd(n')}} + \frac{(n' \cdot m)(n \cdot \mathsf{B}n')}{\gcd(n)\gcd(n')} x^{\mathsf{B}(n+n')} y^{n+n'} \right) (1 + x^{\mathsf{B}n} y^n)^{\frac{n \cdot m}{\gcd(n)}} x^m \\
&= \left((1 + x^{\mathsf{B}n'} y^{n'})^{\frac{n' \cdot m}{\gcd(n')}} (1 + x^{\mathsf{B}n} y^n)^{\frac{n \cdot m}{\gcd(n)}} + \frac{(n' \cdot m)(n \cdot \mathsf{B}n')}{\gcd(n)\gcd(n')} x^{\mathsf{B}(n+n')} y^{n+n'} \right) x^m
\end{aligned}$$

For comparison, $E_{n'}E^{\lambda}_{n+n'}E_n(x^m)$ is equal to

$$\begin{aligned}
&= E_{n'}E^{\lambda}_{n+n'}\left((1+x^{\mathsf{B}n}y^n)^{\frac{n\cdot m}{\gcd(n)}}x^m\right)\\
&= E_{n'}\left((1+x^{\mathsf{B}n}y^n)^{\frac{n\cdot m}{\gcd(n)}}(1+x^{\mathsf{B}(n+n')}y^{n+n'})^{\lambda\frac{(n+n')\cdot m}{\gcd(n+n')}}x^m\right)\\
&= E_{n'}\left(((1+x^{\mathsf{B}n}y^n)^{\frac{n\cdot m}{\gcd(n)}}+\lambda\frac{(n+n')\cdot m}{\gcd(n+n')}x^{\mathsf{B}(n+n')}y^{n+n'})x^m\right)\\
&= E_{n'}E_n(x^m)+\left(\lambda\frac{(n+n')\cdot m}{\gcd(n+n')}x^{\mathsf{B}(n+n')}y^{n+n'}\right)x^m
\end{aligned}$$

We may use the previous computation to evaluate $E_{n'}E_n(x^m)$, and plug in.

$$=\left((1+x^{\mathsf{B}n}y^n)^{\frac{n\cdot m}{\gcd(n)}}(1+x^{\mathsf{B}n'}y^{n'})^{\frac{n'\cdot m}{\gcd(n')}}-\frac{(n\cdot m)(n'\cdot \mathsf{B}n)}{\gcd(n)\gcd(n')}x^{\mathsf{B}(n+n')}y^{n+n'}\right)x^m$$

Since the two sides coincide on a generating set for $\widehat{\mathcal{F}}/I$, the morphisms coincide.

Case (a) follows from Case (b), because $E_{n+n'}$ is trivial mod I if $y^{n+n'}\in I$. □

1.3. **The pronilpotent groups $\widehat{\mathbb{E}}(\mathsf{B})$ and $\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B})$.** Let $\mathbb{E}(\mathsf{B})$ and $\mathbb{E}^{\mathbb{Q}}(\mathsf{B})$ denote the subgroups of $Aut(\widehat{\mathcal{F}}(\mathsf{B}))$ generated by all formal elementary transformations, and all fractional powers of formal elementary transformations (respectively). Tautologically, elements of these groups are finite products of the form

$$E^{d_1}_{n_1}E^{d_2}_{n_2}\cdots E^{d_k}_{n_k}$$

where the d_is are integral or fractional, as appropriate.

The topology on $\widehat{\mathcal{F}}(\mathsf{B})$ induces a topology of pointwise-convergence on $Aut(\widehat{\mathcal{F}}(\mathsf{B}))$.

$$\widehat{\mathbb{E}}(\mathsf{B}) := \text{closure of } \mathbb{E}(\mathsf{B})\in Aut(\widehat{\mathcal{F}}(\mathsf{B})),\quad \widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B}) := \text{closure of } \mathbb{E}^{\mathbb{Q}}(\mathsf{B})\in Aut(\mathbb{Q}\otimes\widehat{\mathcal{F}}(\mathsf{B}))$$

Naturally, we have an inclusion $\widehat{\mathbb{E}}(\mathsf{B})\subset\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B})$.

What do elements of $\widehat{\mathbb{E}}(\mathsf{B})$ look like? The group $\mathbb{E}(\mathsf{B})$ fixes every monomial ideal $I\subset\mathbb{Z}[[y_1,y_2,...,y_r]]$, and so the action on $\widehat{\mathcal{F}}(\mathsf{B})$ descends to an action on $\widehat{\mathcal{F}}(\mathsf{B})/I$. This action is not faithful; let $Im_{\widehat{\mathcal{F}}(\mathsf{B})/I}(\mathbb{E}(\mathsf{B}))$ denote the image of this action in $Aut(\widehat{\mathcal{F}}(\mathsf{B})/I)$. For any pair of monomial ideals $I_1\subset I_2$, there is is a quotient map

$$Im_{\widehat{\mathcal{F}}(\mathsf{B})/I_1}(\mathbb{E}(\mathsf{B}))\longrightarrow Im_{\widehat{\mathcal{F}}(\mathsf{B})/I_2}(\mathbb{E}(\mathsf{B}))$$

Proposition 1.3.1. *The group $\widehat{\mathbb{E}}(\mathsf{B})$ is the inverse limit of $Im_{\widehat{\mathcal{F}}(\mathsf{B})/I}(\mathbb{E}(\mathsf{B}))$, taken over all monomial ideals I with finite codimension.*[2]

Similarly, $\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B})$ is the inverse limit of the images of $\mathbb{E}^{\mathbb{Q}}(\mathsf{B})$ in $Aut(\mathbb{Q}\otimes\mathcal{F}(B)/I)$.

This indicates the following description of elements of $\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B})$ as certain infinite products. Consider a totally ordered set $(S,\prec)$, a finite-to-one map $n:S\to\mathbb{N}^r$, and a map $d:S\to\mathbb{Q}$.

[2]Note that a monomial ideal I has finite codimension if and only if it is an open subset of $\widehat{\mathcal{F}}(\mathsf{B})$.

For any monomial ideal $I \subset \mathbb{Z}[[y_1, y_2, ..., y_r]]$ with finite codimension, the subset

$$S_I = \{s \in S \mid y^{n(s)} \notin I\}$$

is finite. Define the **ordered product mod** I to be the automorphism of $\widehat{\mathcal{F}}(\mathsf{B})/I$

$$\prod_{s \in S_I} E_{n(s)}^{d(s)}$$

where the product is ordered so that $s_1 \prec s_2$ implies that $E_{n(s_2)}$ appears to the left of $E_{n(s_1)}$. Define the **ordered product** to be the automorphism of $\widehat{\mathcal{F}}(\mathsf{B})$

$$\prod_{s \in S} E_{n(s)}^{d(s)}$$

given by the inverse limit of the ordered products mod I, as I runs over all monomial ideals of finite codimension.

Proposition 1.3.2. *The group* $\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B})$ *is the set of ordered products of fractional powers of formal elementary transformations (as automorphisms of* $\mathbb{Q} \otimes \widehat{\mathcal{F}}(\mathsf{B})$*), and* $\widehat{\mathbb{E}}(\mathsf{B})$ *is the subgroup of ordered products with integral powers.*

Remark 1.3.3. Can $\widehat{\mathbb{E}}(\mathsf{B})$ or $\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B})$ be characterized as the group of automorphism of $\widehat{\mathcal{F}}(\mathsf{B})$ or $\mathbb{Q} \otimes \widehat{\mathcal{F}}(\mathsf{B})$ which preserve some additional structure? As a non-example, the matrix B determines a Poisson structure on $\widehat{\mathcal{F}}(\mathsf{B})$ (see Section **??**), and elements of $\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B})$ preserve this structure, but not every Poisson automorphism is of this form.

Infinite products allow us to consider many more interest expressions involving formal elementary transformations. The following identity is fundamental, though we lack the techniques to prove it at the moment.

Proposition 1.3.4. *Let* $n_1, n_2 \in \mathbb{N}^r$ *such that* $n \cdot \mathsf{B} n' = 2$ *and such that* $\gcd(an_1 + bn_2) = \gcd(a, b)$ *for all* $a, b \in \mathbb{N}$*. Then*

$$E_{n_1} E_{n_2} = E_{n_2} E_{n_1+2n_2} E_{2n_1+3n_2} \cdots \left(\prod_{k=0}^{\infty} E_{2^k(n_1+n_2)} \right)^2 \cdots E_{3n_1+2n_2} E_{2n_1+n_2} E_{n_1}$$

Exercise 1.3. For $n \in \mathbb{N}^r$, show that the infinite product

$$E_n E_{2n} E_{4n} E_{8n} \cdots E_{2^k n} \cdots$$

converges to the automorphism which sends x^m to

$$(1 - x^{\mathsf{B}n} y^n)^{-\frac{n \cdot m}{\gcd(n)}} x^m$$

Exercise 1.4. For $n \in \mathbb{N}^r$ and $f \in \mathbb{Z}[x^{\pm \mathsf{B}n}][[y^n]]$, consider $\phi : \widehat{\mathcal{F}}(\mathsf{B}) \to \widehat{\mathcal{F}}(\mathsf{B})$ given by

$$\phi(x^m) = f^{n \cdot m} x^m, \quad \phi(y^{n'}) = y^{n'}$$

Show that ϕ is an element of $\widehat{\mathbb{E}}(\mathsf{B})$.

2. Elementary transformations: topics

2.1. Rational elementary transformations. The y-variables in $\widehat{\mathcal{F}}(\mathsf{B})$ may be regarded as formal place-holders which allow us to make sense of infinite series and compositions without worrying about convergence issues. If we are feeling bold, we may remove them.

Given an $r \times r$ skew-symmetric B, define

$$\mathcal{F}(\mathsf{B}) := \mathbb{Q}(x_1, x_2, ..., x_r)$$

and for any non-zero $n \in \mathbb{N}^r$, define the **rational elementary transformation**

$$E_{n,\mathsf{B}}(x^m) = (1 + x^{\mathsf{B}n})^{\frac{n \cdot m}{\gcd(n)}} x^m, \quad E_{n,\mathsf{B}}(y^{n'}) = y^{n'}$$

Note that a rational elementary transformation E_n is trivial if $\mathsf{B}n = 0$.

The two types of elementary transformation may be connected as follows. A formal elementary transformation E_n preserves the subring of $\widehat{\mathcal{F}}(\mathsf{B})$ consisting of formal series with a rational representation f/g, such that the denominator g is not zero modulo $\langle y_1, ..., y_r\rangle$, nor is it zero modulo $\langle y_1 - 1, ..., y_r - 1\rangle$. This subring then has a map to $\mathcal{F}$ which sends x_i to x_i and y_i to 1, and this map commutes with respective actions of E_n.

That is, we have the following diagram of rings.

$$\begin{tikzcd}
& \left\{ f/g \mid f, g \in \mathbb{Z}[x_1^{\pm 1}, ..., x_r^{\pm 1}, y_1, ..., y_r], g \notin \langle y_1, ..., y_r\rangle, g \notin \langle y_1 - 1, ..., y_r - 1\rangle \right\} \arrow[dl, bend right, "\text{Inclusion}"'] \arrow[dr, bend left, "y_i \mapsto 1"] & \\
\widehat{\mathcal{F}}(\mathsf{B}) = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_r^{\pm 1}][[y_1, y_2, ..., y_r]] & & \mathcal{F}(\mathsf{B}) = \mathbb{Q}(x_1, x_2, ..., x_r)
\end{tikzcd}$$

For $n \in \mathbb{N}^r$, there is an action of E_n on each of these three rings which commutes with the two maps.

Any finite product of formal elementary transformations also determines a finite product of rational elementary transformations which makes the above diagram commute, and so $\mathbb{E}(\mathsf{B})$ has a well-defined action on $\mathcal{F}(\mathsf{B})$. However, an infinite ordered product of formal elementary transformations may not preserve the subring in the diagram, and so $\widehat{\mathbb{E}}(\mathsf{B})$ does not have a natural action on $\mathcal{F}(\mathsf{B})$. This is one of the main justifications for working with formal elementary transformations instead of rational ones.

Remark 2.1.1. The elements of $\widehat{\mathbb{E}}(\mathsf{B})$ arising from scattering diagrams of cluster algebras (see Section **??**) may always determine well-defined automorphisms of $\mathcal{F}(\mathsf{B})$. We don't know, and would very much like to know.

Example 2.1.2. For any non-zero $n \in \mathbb{N}^r$, the product of rational elementary transformations

$$E_n E_{2n} E_{4n} \cdots E_{2^k n} \cdots$$

converges to a well-defined automorphism of $\mathcal{F}(\mathsf{B})$ (see Exercise **??**), but

$$E_n E_{2n}^2 E_{4n}^4 \cdots E_{2^k n}^{2^k} \cdots$$

does not.

2.2. **Poisson structures.** Given an $r \times r$ skew-symmetric matrix B, the associated ring $\widehat{\mathcal{F}}$ has a Poisson structure defined as follows.

$$\{x^{m_1}, x^{m_2}\} = 0, \quad \{x^m, y^n\} = (n \cdot m) x^m y^n, \quad \{y^{n_1}, y^{n_2}\} = (n_1 \cdot \mathsf{B} n_2) y^{n_1 + n_2}$$

Proposition 2.2.1. *Elementary transformations are Poisson automorphisms of $\widehat{\mathcal{F}}$:*

$$E_n(\{f, g\}) = \{E_n(f), E_n(g)\}$$

This is not an accident; the elementary transformations can actually be reconstructed from the Poisson structure in characteristic zero. For any $f \in \widehat{\mathcal{F}}(\mathsf{B})$ in the ideal generated by the y-variables, define the **dilogarithm**

$$Li_2(f) := \sum_{k=1}^{\infty} \frac{f^k}{k^2} \in \mathbb{Q} \otimes \widehat{\mathcal{F}}$$

Proposition 2.2.2. *For all n, m,*

$$\{-Li_2(-x^{\mathsf{B}n} y^n), x^m\} = (n \cdot m) \log(1 + x^{\mathsf{B}n} y^n) x^m \text{ and } \{-Li_2(-x^{\mathsf{B}n} y^n), y^{n'}\} = 0$$

Hence, E_n^d is the exponential of the differential operator $\frac{d}{\gcd(n)}\{-Li_2(-x^{\mathsf{B}n} y^n), -\}$.

Remark 2.2.3. The proposition gives an geometric interpretation of E_n^d, as the time $\frac{d}{\gcd(n)}$ flow along the Hamiltonian vector field of the function $-Li_2(-x^{\mathsf{B}n} y^n)$ on $Spf(\widehat{\mathcal{F}})$.

2.3. **Compatible inclusions.** Fix skew-symmetric integer matrices B_1 and B_2. A **compatible inclusion** from $\widehat{\mathcal{F}}(\mathsf{B}_1) \rightarrow \widehat{\mathcal{F}}(\mathsf{B}_2)$ is a ring map of the form

$$\phi(x^m) = x^{\mathsf{A}m}, \quad \phi(y^n) = x^{(\mathsf{B}_2 \mathsf{C} - \mathsf{A} \mathsf{B}_1) n} y^{\mathsf{C}n}$$

where A and C are $|\mathsf{B}_2| \times |\mathsf{B}_1|$ integer matrices, such that C has non-negative entries and there is a non-zero integer λ with

$$\mathsf{A}^\top \mathsf{C} = \lambda \cdot \mathsf{Id}, \quad \mathsf{C}^\top \mathsf{B}_2 \mathsf{C} = \lambda \cdot \mathsf{B}_1$$

Note that the first equation implies that A and C are inclusions, and so ϕ is also an inclusion.

For a non-zero integer λ, a morphism ϕ of Poisson algebras is **λ-Poisson** if

$$\lambda \phi(\{-, -\}) = \{\phi(-), \phi(-)\}$$

Proposition 2.3.1. *A ring map $\phi : \widehat{\mathcal{F}}(\mathsf{B}_1) \rightarrow \widehat{\mathcal{F}}(\mathsf{B}_2)$ is a compatible inclusion if and only if it is a monomial λ-Poisson morphism which sends elements of the form $x^{\mathsf{B}_1 n} y^n$ to elements of the form $x^{\mathsf{B}_2 n'} y^{n'}$.*

Proposition 2.3.2. *Given a compatible inclusion $\phi : \widehat{\mathcal{F}}(\mathsf{B}_1) \hookrightarrow \widehat{\mathcal{F}}(\mathsf{B}_2)$, there is an inclusion of topological groups*

$$\Phi : \widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B}_1) \hookrightarrow \widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B}_2), \quad \Phi(E^d_{\mathsf{B}_1,n}) = E^{\frac{d \gcd(\mathsf{C}n)}{\lambda \gcd(n)}}_{\mathsf{B}_2,\mathsf{C}n}$$

which intertwines the respective action on $\mathbb{Q} \otimes \widehat{\mathcal{F}}(\mathsf{B}_1)$ and $\mathbb{Q} \otimes \widehat{\mathcal{F}}(\mathsf{B}_2)$; that is,

$$\phi \circ E^d_{\mathsf{B}_1,n} = \Phi(E^d_{\mathsf{B}_1,n}) \circ \phi$$

Note that the inclusion Φ only depends on C and λ, and not on A.

The proposition is endlessly useful for describing products in $\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B}_2)$ as potentially simpler products in another group $\widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B}_1)$.

(1) For any B and any $\lambda \in \mathbb{Z}$, there are compatible inclusions

$$\phi : \widehat{\mathcal{F}}(\mathsf{B}) \hookrightarrow \widehat{\mathcal{F}}(\lambda\mathsf{B}), \quad x^m \mapsto x^{\lambda m}, \quad y^n \mapsto y^n$$

$$\Phi : \widehat{\mathbb{E}}^{\mathbb{Q}}(\mathsf{B}) \hookrightarrow \widehat{\mathbb{E}}^{\mathbb{Q}}(\lambda\mathsf{B}), \quad E_{\mathsf{B},n} \mapsto E^{1/\lambda}_{\lambda\mathsf{B},n}$$

(2) For any B and any $\lambda \in \mathbb{Z}$, there are compatible inclusions

$$\phi : \widehat{\mathcal{F}}(\lambda\mathsf{B}) \hookrightarrow \widehat{\mathcal{F}}(\mathsf{B}), \quad x^m \mapsto x^{m}, \quad y^n \mapsto y^{\lambda n}$$

$$\Phi : \widehat{\mathbb{E}}(\lambda\mathsf{B}) \hookrightarrow \widehat{\mathbb{E}}(\mathsf{B}), \quad E_{\lambda\mathsf{B},n} \mapsto E_{\mathsf{B},\lambda n}$$

(3) For any B and any $r' < r$, let B' be the upper left $r' \times r'$ submatrix of B and F the lower left $(r-r') \times r'$ submatrix of B. Then there are compatible inclusions

$$\phi : \widehat{\mathcal{F}}(\mathsf{B}') \hookrightarrow \widehat{\mathcal{F}}(\mathsf{B}), \quad x^m \mapsto x^{(m,0)}, \quad y^n \mapsto x^{(0,\mathsf{F}n)} y^{(n,0)}$$

$$\Phi : \widehat{\mathbb{E}}(\mathsf{B}') \hookrightarrow \widehat{\mathbb{E}}(\mathsf{B}), \quad E_{\mathsf{B}',n} \mapsto E_{\mathsf{B},(n,0)}$$

Exercise 2.1. Let $n, n' \in \mathbb{N}^r$ such that $n \cdot \mathsf{B}n' = 1$. Use Exercise **??** to show that

(1) $E_n E_{2n'} = E_{2n'} E_{n+2n'} E_{2n+2n'} E_n$
(2) $E_n E_{3n'} = E_{3n'} E_{n+3n'} E_{3n+6n'} E_{2n+3n'} E_{3n+3n'} E_n$

2.4. **The tropical vertex group.** One of the most important applications of Proposition [?] is to reduce any computation involving a pair of elementary transformations to a computation inside a fixed group $\widehat{\mathbb{E}}(\mathsf{J})$. Define

$$\mathsf{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For any B and any n_1, n_2 such that $\lambda := n_1 \cdot \mathsf{B}n_2 \neq 0$, there are compatible inclusions

$$\phi : \widehat{\mathcal{F}}(\mathsf{J}) \hookrightarrow \widehat{\mathcal{F}}(\mathsf{B}), \quad x_1 \mapsto x^{\mathsf{B}n_2}, \quad x_2 \mapsto x^{-\mathsf{B}n_1}, \quad y_1 \mapsto y^{n_1}, \quad y_2 \mapsto y^{n_2}$$

$$\Phi : \widehat{\mathbb{E}}(\mathsf{J}) \hookrightarrow \widehat{\mathbb{E}}(\mathsf{B}), \quad E_{\mathsf{J},(1,0)} \mapsto E^{\gcd(n_1)/\lambda}_{\mathsf{B},n_1}, \quad E_{\mathsf{J},(0,1)} \mapsto E^{\gcd(n_2)/\lambda}_{\mathsf{B},n_2}$$

As a consequence, for any n_1, n_2 such that $n_1 \cdot \mathsf{B}n_2 \neq 0$, we have

$$E^b_{\mathsf{B},n_1} E^c_{\mathsf{B},n_2} = \Phi(E^{\lambda b/\gcd(n_1)}_{\mathsf{J},(1,0)} E^{\lambda c/\gcd(n_2)}_{\mathsf{J},(0,1)})$$

This means that relations involving pairs of formal elementary transformations can all be reduced to relations in $\widehat{\mathbb{E}}(\mathsf{J})$. The group $\widehat{\mathbb{E}}(\mathsf{J})$ was introduced in [**?**] who called it the *tropical vertex group.*

Example 2.4.1. Proposition [**?**] is equivalent to the following identity in $\widehat{\mathbb{E}}(\mathsf{J})$.

$$E_{(1,0)}^2 E_{(0,1)}^2 = E_{(0,1)}^2 E_{(1,2)}^2 E_{(2,3)}^2 \cdots \left(\prod_{k=0}^{\infty} E_{(2^k,2^k)} \right)^4 \cdots E_{(3,2)}^2 E_{(2,1)}^2 E_{(1,0)}^2$$

2.5. **Decompositions.** For any $n_0 \in \mathbb{N}^r$, define several subgroups.

$$\widehat{\mathbb{E}}_+^{n_0}(\mathsf{B}) := \text{closed subgroup gen. by } \{E_n \text{ such that } n \cdot \mathsf{B} n_0 > 0\}$$

$$\widehat{\mathbb{E}}_-^{n_0}(\mathsf{B}) := \text{closed subgroup gen. by } \{E_n \text{ such that } n \cdot \mathsf{B} n_0 < 0\}$$

$$\widehat{\mathbb{E}}_{||}^{n_0}(\mathsf{B}) := \text{closed subgroup gen. by } \{E_n \text{ such that } n_0 \in \mathbb{Q}n\}$$

$$\widehat{\mathbb{E}}_\perp^{n_0}(\mathsf{B}) := \text{closed subgroup gen. by } \{E_n \text{ such that } n \cdot \mathsf{B} n_0 = 0 \text{ and } n_0 \notin \mathbb{Q}n\}$$

Remark 2.5.1. Important caveat! The definition of this decomposition appears to reverse the order of n and n_0 compared to [GHKK]. This is actually to compensate for the fact that their skew-form corresponds to $-\mathsf{B}$, not B.

Proposition 2.5.2. *For every $n_0 \in \mathbb{N}^r$, there is a decomposition of groups*

$$\widehat{\mathbb{E}}(\mathsf{B}) = \widehat{\mathbb{E}}_+^{n_0}(\mathsf{B}) \cdot (\widehat{\mathbb{E}}_{||}^{n_0}(\mathsf{B}) \times \widehat{\mathbb{E}}_\perp^{n_0}(\mathsf{B})) \cdot \widehat{\mathbb{E}}_-^{n_0}(\mathsf{B})$$

Let $\Psi_{n_0} : \widehat{\mathbb{E}}(\mathsf{B}) \to \widehat{\mathbb{E}}_{||}^{n_0}(\mathsf{B})$ be the projection onto the middle factor.

Proposition 2.5.3. *The map*

$$\left(\prod_{n_0 \ \textit{principle}} \Psi_{n_0} \right) : \widehat{\mathbb{E}}(\mathsf{B}) \longrightarrow \prod_{n_0 \ \textit{principle}} \widehat{\mathbb{E}}_{||}^{n_0}(\mathsf{B})$$

is a set bijection.

Example 2.5.4. Let $\mathsf{B} = \mathsf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and let

$$G = E_{(1,0)} E_{(0,1)} = E_{(0,1)} E_{(1,1)} E_{(0,1)}$$

If $n_0 = (1,0)$, then

$$G = \underbrace{E_{(1,0)}}_{G_{||}} \underbrace{E_{(0,1)}}_{G_-}, \quad \Psi_{(1,0)}(G) = E_{(1,0)}$$

If $n_0 = (1,1)$, then

$$G = \underbrace{E_{(1,0)}}_{G_+} \underbrace{E_{(0,1)}}_{G_-}, \quad \Psi_{(1,1)}(G) = 1$$

If $n_0 = (0, 1)$, then

$$G = \underbrace{E_{(1,0)}}_{G_+} \underbrace{E_{(0,1)}}_{G_{||}}, \quad \Psi_{(0,1)}(G) = E_{(0,1)}$$

All other $\Psi_{n_0}(G)$ are trivial.

Example 2.5.5. Let $\mathsf{B} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, and let

$$G = E_{(0,1,0)} E_{(1,1,0)} E_{(0,0,1)} E_{(1,0,0)}$$

If $n_0 = (1, 0, 0)$, then

$$G = \underbrace{E_{(0,0,1)}}_{G_+} \underbrace{E_{(1,0,0)}}_{G_{||}} \underbrace{E_{(0,1,1)}}_{G_\perp} \underbrace{E_{(0,1,0)}}_{G_-}, \quad \Psi_{(0,1,1)}(G) = E_{(1,0,0)}$$

If $n_0 = (0, 1, 1)$, then

$$G = \underbrace{E_{(0,1,0)} E_{(1,1,0)}}_{G_+} \underbrace{E_{(1,0,0)}}_{G_\perp} \underbrace{E_{(1,0,1)} E_{(0,0,1)}}_{G_-}, \quad \Psi_{(1,0,0)}(G) = 1$$

3. Scattering diagrams

A scattering diagram may be regarded as a method for visualizing diagrams among formal elementary transformations (including infinite products).

3.1. **Scattering diagrams.** A **linear polyhedral cone** in $\mathbb{R}^r$ is a subset given by a finite intersection of half-spaces

$$\bigcap_{\text{finite}} \{v \in \mathbb{R}^r \mid \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r \geq 0\}$$

Note that such a cone may be supported on a lower dimensional subspace (or even just the origin). An **affine polyhedral cone** in $\mathbb{R}^r$ is an affine translation of a linear polyhedral cone.

For a skew-symmetric matrix B,[3] an **elementary wall** is a pair (n, W), of

- a non-zero element $n \in \mathbb{N}^r$, and
- an affine polyhedral cone W in $\mathbb{R}^r$, whose affine span is $w + n^\perp := \{w + m \in \mathbb{R}^r \mid n \cdot m = 0\}$ for any $w \in W$.

A wall (n, W) is **linear** (resp. a **hyperplane wall**) if W is a linear polyhedral cone (resp. an affine hyperplane).

Notice that W *almost* determines n; the set of $n \in \mathbb{N}^r$ normal to W is a semigroup isomorphic to $\mathbb{N}$.[4] In particular, n is determined by W and the value of $\gcd(n)$. A wall (n, W) will be called **principal** if $\gcd(n) = 1$; equivalently, if it generates the set of non-negative integral normal vectors.

The normal vector n distinguishes between the two sides of the hyperplane $affspan(W)$. We will refer to the side in which n points as the **green** side of W, and the other side as the **red** side of W.

Definition 3.1.1. For a skew-symmetric matrix B, a **scattering diagram** $\mathfrak{D}$ is a set of walls of B, each with a **multiplicity** in $\mathbb{Q}$, such that for any $n \in \mathbb{N}^r$, there are at most finitely-many walls of the form $(n, *)$.

We specifically allow infinitely many walls of the form $(*, W)$ for the same W. A **linear** scattering diagram is one in which every wall is linear.

3.2. **Transverse paths and path-ordered products.** A **transverse path** p in $\mathfrak{D}$ is a piecewise-smooth[5] path $p : [0, 1] \to \mathbb{R}^r$ such that

- $p(0)$ and $p(1)$ are not in any walls,
- whenever the image of p intersects a wall, it crosses it transversely, and

[3] While B does not appear in the definition of a wall, it is necessary to associate a formal elementary transformation E_n to the wall, and so we shouldn't consider walls without first having a matrix in mind.

[4] For general affine polyhedral cone, there may be no non-zero elements in $\mathbb{N}^r$ normal to W, but we won't consider potential walls of this form.

[5] It would be useful to weaken this as much as possible.

- the image of p does not intersect the boundary of a wall or the intersection of two walls which span different hyperplanes.[6]

A transverse path is **finite** if its image intersects finitely many walls in $\mathfrak{D}$.

A finite transverse path p determines a finite product of formal elementary transformations of $\widehat{\mathcal{F}}$, as follows. List the walls crossed by p in order:[7]

$$d_1 \cdot (n_1, W_1), d_2 \cdot (n_2, W_2), ..., d_k \cdot (n_k, W_k)$$

and to each, associate the rational number

$$\epsilon_i := \left\{ \begin{array}{ll} +d_i & \text{if } p \text{ crossed } W_i \text{ from the green side to the red side} \\ -d_i & \text{if } p \text{ crossed } W_i \text{ from the red side to the green side} \end{array} \right\}$$

Then the **path-ordered product** of p is

$$E_{n_k}^{\epsilon_k} E_{n_{k-1}}^{\epsilon_{k-1}} \cdots E_{n_1}^{\epsilon_1}$$

Notice that we use the 'function composition' ordering, and not the left-to-right ordering.

In the same way, an infinite transverse path p also determines a (infinite) path-ordered product of formal elementary transformations. By the finiteness condition in the definition of scattering diagrams, this infinite product converges in $\widehat{\mathbb{E}}(\mathsf{B})$.

Two scattering diagrams $\mathfrak{D}_1$ and $\mathfrak{D}_2$ for the same B are **equivalent** if, for each path P which is transverse in both scattering diagrams, the path-ordered products are the same.

The path-ordered product only depends on the transverse path up to homotopy.

Proposition 3.2.1. *Given a scattering diagram $\mathfrak{D}$, let $P : [0,1] \times [0,1] \to \mathbb{R}^r$ be a smooth map such that $P(t, -)$ is a transverse path for all $t \in [0,1]$. Then the path-ordered products of $P(0,-)$ and $P(1,-)$ coincide.*

We say two transverse paths are **equivalent** if they may be realized as $P(0,-)$ and $P(1,-)$ for such a map.

Definition 3.2.2. A scattering diagram is **consistent** if, for each pair of transverse paths with the same endpoints, the associated path ordered products are equal.

Of course, checking every pair of paths is unnecessary.

Lemma 3.2.3. *A scattering diagram $\mathfrak{D}$ is consistent if there is a locally finite set of open sets $\{U_i \subset \mathbb{R}^r\}$ such that*

- *the union $\bigcup U_i$ is dense and simply connected, and*
- *any transverse loop contained in any U_i has trivial path-ordered product.*

In particular, it suffices to check every 'sufficiently small' loop.

[6]This may be weakened to requiring that the normal vectors of the walls are multiples of each other, without changing the consequences.

[7]The path p may simultaneously cross multiple walls, but only if those walls span the same hyperplane. In this case, the walls may be listed in any order, since the corresponding automorphisms commute.

3.3. **Reduction of scattering diagrams.** The scattering diagrams we are interested typically have infinitely many walls, and so there are not enough finite transverse loops to test for the 'right' notion of consistency. The simplest solution is to make sense of formal limits of consistent finite scattering diagrams.

Let I be a monomial ideal in $\mathbb{Z}[[y_1, y_2, ..., y_r]]$. For each $n \in \mathbb{N}^r$ and $d \in \mathbb{Q}$, E_n^d descends to a well-defined automorphism of $\widehat{\mathcal{F}}/I$, which is trivial when $y^n \in I$. A finite scattering diagram is **consistent mod** I if the path-ordered product associated to every transverse loop is the trivial automorphism of $\widehat{\mathcal{F}}/I$.

The **reduction** $\mathfrak{D}/I$ of a scattering diagram $\mathfrak{D}$ is obtained by deleting every wall of the form (n, W) with $y^n \in I$. Clearly, if $\mathfrak{D}$ is a consistent finite scattering diagram, then $\mathfrak{D}/I$ is consistent mod I. This idea can be extended to a criterion for the consistency of infinite scattering diagrams.

Proposition 3.3.1. *A scattering diagram $\mathfrak{D}$ is consistent if and only if, for each monomial ideal $I \subset \mathbb{Q}[[y_1, y_2, ..., y_r]]$ with finite dimensional quotient, the reduction $\mathfrak{D}/I$ is finite and consistent mod I.*

Exercise 3.1. Let $\mathfrak{m} := \langle y_1, y_2, ..., y_r \rangle$. Prove that a hyperplane scattering diagram (one consisting of hyperplane walls) is consistent mod $\mathfrak{m}^2$.

Exercise 3.2. Let $\mathfrak{m} := \langle y_1, y_2, ..., y_r \rangle$. Prove that a scattering diagram is consistent if and only if it is consistent mod $\mathfrak{m}^i$ for all i.

3.4. **Linearizing scattering diagrams.** Given a scattering diagram $\mathfrak{D}$, its **linearization** is the scattering diagram which replaces each wall in $\mathfrak{D}$ with the corresponding linear wall.

Proposition 3.4.1. *If $\mathfrak{D}$ is consistent (resp. consistent mod I), then its linearization is consistent (resp. consistent mod I).*

3.5. **Consistent extension.** One of the fundamental results regarding scattering diagrams is that any scattering has an essentially unique 'minimal' completion to a consistent scattering diagram. A wall (n, W) is **outgoing** if $w - \mathbb{R}_{\geq 0}\mathsf{B}n \not\in W$ for every $w \in W$.

Theorem 3.5.1. *Let $\mathfrak{D}_{in}$ be a scattering diagram. Then there is a consistent scattering diagram $\mathfrak{D}_{out}$ such that $\mathfrak{D}_{out} - \mathfrak{D}_{in}$ consists of outgoing walls, and this $\mathfrak{D}_{out}$ is unique up to equivalence.*

If the multiplicities in $\mathfrak{D}_{in}$ are positive integral, and there is a monomial ideal I such that $\mathfrak{D}_{in}/I = \mathfrak{D}_{in}$ and $\mathfrak{D}_{in}$ is consistent mod I, then all the multiplicities in $\mathfrak{D}_{out}$ are positive integral.

Call $\mathfrak{D}_{out}$ the **consistent completion** of $\mathfrak{D}_{in}$.

We will most often apply the theorem to hyperplane scattering diagrams $\mathfrak{D}_{in}$ with positive integral multiplicities. In this case, $\mathfrak{D}_{in}/\mathfrak{m}^2 = \mathfrak{D}_{in}$ and $\mathfrak{D}_{in}$ is consistent mod $\mathfrak{m}^2$, so the second half of the theorem implies that $\mathfrak{D}_{out}$ has positive integral multiplicities.

Exercise 3.3. Prove that consistent completion commutes with linearization.

3.6. **Scattering diagrams of elements in $\widehat{\mathbb{E}}$.** A **chamber** in a scattering diagram is a path-connected component of the complement of the walls. In a consistent scattering diagram, there is at most one chamber which is on the green side (resp. red side) of every wall; call this the **green chamber** (resp. the **red chamber**). If $\mathfrak{D}$ is linear, then the green chamber contains all-positive orthant $\mathbb{R}_{>0}$ and the red chamber contains the all-negative orthant (and both chambers always exist).

Given a consistent scattering diagram $\mathfrak{D}$ for B, any pair of chambers C_1, C_2 determine a well-defined element of $\widehat{\mathbb{E}}(\mathsf{B})$, given by the path-ordered product of any transitive path from C_1 to C_2. The **characteristic automorphism** of a consistent scattering diagram $\mathfrak{D}$ is the element $E_{+,-}(\mathfrak{D}) \in \widehat{\mathbb{E}}(\mathsf{B})$ given the path-ordered product of any transitive path from the green chamber to the red chamber.[8]

Theorem 3.6.1 (GHKK)**.** *The map of sets*

$$E_{+,-} : \{\textit{consistent linear scattering diagrams for } \mathsf{B}\}/\textit{equiv.} \longrightarrow \widehat{\mathbb{E}}(\mathsf{B})$$

is a bijection.

From the perspective of this theorem, a consistent linear scattering diagram is essentially a visualization of many different factorizations of $E_{+,-}(\mathfrak{D})$ into a (possibly infinite) product of formal elementary transformations.

We may construct the inverse to $E+,-$ as follows. Given $g \in \widehat{\mathbb{E}}(\mathsf{B})$, define the hyperplane scattering diagram

$$\mathfrak{D}(g) : \{\lambda \cdot (n, n^{\perp}) \mid \Psi_n(g) = E_n^{\lambda} \cdot (\text{stuff involving } E_{\alpha n} \text{ for } \alpha \neq 1)\}$$

Theorem 3.6.2. *There is a commutative diagram*

$$\begin{tikzcd}
\widehat{\mathbb{E}}(\mathsf{B}) \arrow[r, bend left, "\mathfrak{D}(-)"] & \{\textit{Hyperplane scat. diag. for } \mathsf{B}\} \arrow[dl, bend left, "\textit{Consistent completion}"] \\
\{\textit{Consistent scat. diag. for } \mathsf{B}\}/\textit{ equiv.} \arrow[u, bend left, "E_{+,-}"] &
\end{tikzcd}$$

[8]In the rare case that there is no green or red chamber, the element $\Theta_{+,-}(\mathfrak{D})$ may be defined as the limit of the characteristic automorphisms of all finite reductions.

Remark 3.6.3. The key idea in the proof is that the element $\Psi_n(E_{+,-})$ is the path-ordered automorphism corresponding to the path $-tn - \epsilon \mathsf{B} n$, where $t \in \mathbb{R}$ stays near 0, and ϵ is small and contained in $n^\perp$.[9]

4. Connection with cluster algebras

4.1. A-type scattering diagrams (simply laced). Let B be a skew-symmetric $r \times r$ matrix. Construct a scattering diagram $\mathfrak{D}_{in}$ consisting of principal walls supported on each coordinate hyperplane.

$$\mathfrak{D}_{in} = \sum_{i=1}^{r} (e_i, e_i^\perp)$$

Definition 4.1.1. The **A-type scattering diagram** $\mathfrak{D}(\mathsf{B})$ associated to B is the consistent completion of $\mathfrak{D}_{in}$ as above.

Remarkably, virtually every algebraic feature of the cluster algebra $\mathcal{A}(\mathsf{B})$ can be interpreted as a geometric feature of $\mathfrak{D}(\mathsf{B})$.

Let $\mathbf{x}$ be a cluster in $\mathcal{A}(\mathsf{B})$. The explicit choice of an initial seed means every cluster variable $x_i \in \mathbf{x}$ has a well-defined **g-vector**, and that the g-vectors of $\mathbf{x}$ are a spanning set of $\mathbb{Z}^r$. The g-vectors of $\mathbf{x}$ generate a simplicial cone in $\mathbb{R}^r$.

A chamber is **reachable** if there is a finite transitive path from the green chamber.

Theorem 4.1.2 (GHKK). *For each cluster $\mathbf{x}$ in $\mathcal{A}(\mathsf{B})$, the cone in $\mathbb{R}^r$ generated by the g-vectors of $\mathbf{x}$ is the closure of a reachable chamber in $\mathfrak{D}(\mathsf{B})$. This determines a bijection of sets*

$$\text{clusters of } \mathcal{A}(\mathsf{B}) \xrightarrow{\sim} \text{reachable chambers in } \mathfrak{D}(\mathsf{B})$$

The cluster monomials in $\mathbf{x}$ also have g-vectors, and the set

4.2. The characteristic automorphism. For any B, let

$$E_{+,-}(\mathsf{B}) := \mathbb{E}(\mathfrak{D}(\mathsf{B})) \in \widehat{\mathbb{E}}(\mathsf{B})$$

Whenever $E_{+,-}(\mathsf{B})$ converges to a rational automorphism of $\mathcal{F}(\mathsf{B})$, call the corresponding automorphism the **non-commutative Donaldson-Thomas series** of B.

Q: When does the element $E_{+,-}(\mathsf{B})$ rationalize?

The easiest way to show this is to know that $E_{+,-}(\mathsf{B}) \in \mathbb{E}(\mathsf{B})$; that is, that it can be written as a finite product of formal elementary transformations.

Proposition 4.2.1. *If there is a finite transverse path in $\mathfrak{D}(\mathsf{B})$ from the positive chamber to the negative chamber, then $E_{+,-}(\mathsf{B})$ rationalizes.*

Such a path is called a **reddening path**, and the corresponding sequence of transformations is called a **reddening sequence**.

[9]Technically, there may be no ϵ small enough. To be precise, we need to take the limit over finite reductions.

A reddening path which only crosses walls from the green side to the red side may be called a **maximal green path**. The corresponding **maximal green sequence** of formal elementary transformations is notable for containing no inverses.

4.3. **Connection with stability conditions.** Let (Q, W) be a quiver with potential, and let B be the adjacency matrix of Q. A **central charge** is a group homomorphism from $Z : \mathbb{Z}^r \to \mathbb{C}$ such that $Im(Z(\mathbb{N}^r)) > 0$. Here, the domain is identified with the Grothendieck group of the $Mod(J(Q, W))$. This map splits into two pieces,

$$Re(Z), Im(Z) : \mathbb{Z}^r \to \mathbb{R}$$

which may be identified with elements of $\mathbb{R}^r$.

Consider the oriented line

$$p_Z(t) = Re(Z) - tIm(Z) \subset \mathbb{R}^r$$

which we think of as a path in the scattering diagram $\mathfrak{D}(\mathsf{B})$. Note that it travels from the all-positive chamber to the all-negative chamber. This lines crosses a wall (n, W) at time t iff

$$Re(Z(n)) = tIm(Z(n)) \Leftrightarrow t = \frac{Re(Z(n))}{Im(Z(n))}$$

The following justifies the terminology *non-commutative Donaldson-Thomas series.*

Theorem 4.3.1. *Assume that the non-commutative Donaldson-Thomas series exists; that is, $E_{+,-}(\mathsf{B})$ rationalizes. The infinitesimal contribution to the path-ordered product of p_Z at time t is equal to the non-commutative Donaldson-Thomas series of the $\mathcal{C}_{t,Z} \subset Nil(Q)$ generated by the Z-semistable objects of slope t.*

As a consequence, the rationalization of $E_{+,-}(\mathfrak{D}(\mathsf{B}))$ is equal to the non-commutative Donaldson-Thomas series of $Nil(Q)$, in the sense of Kontsevich-Soibelman.

Appendix A. Stereographic projection

Stereographic projection may be used to visualize fans in 3-dimensions. The particular stereograhic projection I use may be given as the composition of three operations.

(1) Project $\mathbb{R}^3 \setminus \{0\}$ onto the unit sphere centered at the origin.

$$(x, y, z) \mapsto \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$

(2) Rotate the unit sphere so that the point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ goes to $(0, 0, 1)$. This can be given by the action of the matrix

$$\begin{bmatrix} 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

(3) Stereographically project onto the plane $(*, *, 1)$ from the point $(0, 0, -1)$.

$$(x, y, z) \mapsto \left(\frac{2x}{1-z}, \frac{2y}{1-z} \right)$$

The composition of these three maps sends (g_1, g_2, g_3) to

$$\left(\sqrt{2} \frac{g_3 - g_2}{\sqrt{g_1^2 + g_2^2 + g_3^2} - g_1 - g_2 - g_3}, \sqrt{\frac{2}{3}} \frac{g_2 + g_3 - 2g_1}{\sqrt{g_1^2 + g_2^2 + g_3^2} - g_1 - g_2 - g_3} \right)$$

We can check that this sends $(1, 1, 1)$ to the origin and is undefined for $(-g, -g, -g)$ when $g \geq 0$.

We are also interested in the projection of planes under this map. The plane normal to the vector (c_1, c_2, c_3) is given by

$$\text{center} = \left(\sqrt{6} \frac{c_2 - c_3}{c_1 + c_2 + c_3}, \sqrt{2} \frac{c_2 + c_3 - 2c_1}{c_1 + c_2 + c_3} \right), \quad \text{radius} = 2 \frac{\sqrt{c_1^2 + c_2^3 + c_3^2}}{|c_1 + c_2 + c_3|}$$

We check that the plane normal to $(1, 1, 1)$ goes to the circle centered at the origin with radius $\frac{2}{\sqrt{3}}$. The stereographic projections of the coordinate planes are given by

$$\text{center} = (0, -\sqrt{8}), \quad \text{radius} = 2$$

$$\text{center} = (\sqrt{6}, \sqrt{2}), \quad \text{radius} = 2$$

$$\text{center} = (-\sqrt{6}, \sqrt{2}), \quad \text{radius} = 2$$