Algorithms for Electrical Networks

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This is a wish-list of routines that will facilitate work on inverse problems for electrical networks. More detail will be explained at the blackboard. Also look at the archives of the REU website `http://www.math.washington.edu/~reu/` and the book *Inverse Problems for Electrical Networks*.

1. FORWARD PROBLEM. Compute the response matrix (Dirichlet-to-Neumann map) for an electrical network. An electrical network is a graph (usually not a multigraph) with (arbitrarily) designated boundary vertices and interior vertices, and (usually) with positive real number weights γ_{ij} on the edges. It is denoted by (G, Γ). A $\gamma - harmonic$ function is a vertex function u_i such that $\sum_j \gamma_{ij}(u_i - u_j) = 0$ for all interior vertices i, where the sum is over vertices j adjacent to i. The Dirichlet problem is to find a γ-harmonic function with given boundary values. If the graph is connected there is a unique solution for every choice of boundary function. This solution produces a current flow (positive if into) at the boundary vertices. There is hence a linear map from (the vector) of boundary values (voltages) to a vector of boundary currents. This linear map Λ is the response map. It has lots of nice properties (symmetric, positive semi-definite, row-sum 0, *.etc.*

   If the vertices of the graph are ordered, the information that describes (G, Γ) can be stored in a matrix K, the Kirchhoff matrix. If we list the boundary vertices first and then the interior vertices, K can be partitioned,
   $$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$
   and $\Lambda = A - BC^{-1}B^T$, the Schur complement of C in A, sometimes denoted K/C.

   This is not necessarily the best way to compute Λ. Ordering of the boundary vertices is important. Computing certain subdeterminants is useful.

2. INVERSE PROBLEM Also referred to as *recovery*. Does Λ determine K? If so, compute K from Λ. The map $DN = DN(\gamma)$ from K to Λ a rational function whose components are ratios of polynomials of high degree. So it looks like a problem in algebraic geometry and it might seem that Grobner basis methods would come in handy. So far this is not true. Grobner basis algorithms are too stupid. Most methods need cleverness. However a team of students last summer came up with a very accurate numerical method. It still needed to be clever.

   The question as posed should be sharpened. For example suppose we know G (we know where the location of the 0's in K.) This simplifies the problem and makes it more definite. If the map DN is 1-1 we say G is *recoverable*. If DN is n-1, we say G is n-1. For special classes of graphs, for example *critical circular planar* graphs we know that DN is 1-1. (Define *circular pair, critical, right sign*). Moreover, given a "legal" response matrix and the graph, we have an algorithm to find the entries of K. But what is even better, we can also find up to $Y - \Delta$ equivalence class of the graph itself. This result goes back to Derek Jerina in 1996. He conjectured a "key lemma" that we eventually proved that gave the best method for recovering G. It is referred to in the book *Inverse Problems for Electrical Networks* as the *cut-point lemma*. It somehow says something about the planarity of the graph (it seems reminicent of the Euler-Poincare characteristic) and is phrased in terms of the *medial* graph. What is recovered is the medial graph and this determines the $Y - \Delta$ equivalence class of the graph.

Trouble with the program: it is a clumsily written Mathematica program. See Chris Staskewicz' paper in the 1997 REU archives. Can't "break" it. I'll explain. A well written program would be very useful.

3. MEDIAL GRAPHS A circular planar graph has a well defined medial graph. Place a new vertex on each edge and adjoin vetices if the edges they lie on are adjacent. Add two vertices on the bounding circle between each boundary vertex. This produces a circular planar graph with twice as many boundary vertices and each interior vertex has degree four. The cells are two-colorable. The medial graph encodes the original graph and its dual. A critical circular planar graph, by definition, can be embedded in the unit disc. The medial graph of a graph is determined by and could be defined by an algorithm. Nathan Pflueger defined such a prodedure in 2006. Then one could write a routine to test for criticality, by checking for lenses in the medial graph. A circular planar graph is critical (and hence recoverable) if and only if the medial graph has no lenses. It would also be useful to have a routine that draws the circular embedding and the medial graph. By a theorem of Whitney the embedding is essentially unique (see Eliana Hechter's paper from the 2005 REU; and also Ryan Sturgell in 1998 and Thomas Carlson in 1999).

4. NON-PLANAR GRAPHS The most interesting questions currently involve non-planar graphs. For a variety of reasons it is helpful to find embeddings of graphs on compact Riemann surfaces and to find the minimal genus of such an embedding. There is a slow algorithm that determines the genus, essentially by producing all possible embeddings. The idea is to consider all circular orderings of edges at a vertex of a graph and then construct faces compatible with a given ordering. For each ordering there results a list of faces and adjacent edges and vertices. In 2005, Owen Biesel, Jeff Eaton, and Orion Bawdon wrote a Matlab program to do this. A similar program was written in 2004 by Nick Reichert. Orion wrote a Java program with improvements, which has since been lost and not replaced. It would be great to have a program that at least runs on graphs with a small number of vertices (maybe less than 100??). Related to this is the *circular embedding* of a graph – an embedding of a graph in a compact Riemann surface with the boundary vertices in given order on the boundary of a disk and all other vertices outside of the disk. This would be most useful in coming up with a definition of medial graph of an graph that is not circular planar.

5. GENERAL RECOVERY ROUTINES Several students have produced routines that "solve" the inverse problem for special non-planar graphs (see Jenny French and Jerry Pan in 2004,

Ilya Grigoriev in 2006 found a long-sought 3-1 graph. These routines are based on the ideas of Jeff Russell and Nick Addington in 2004 to think of DN as a composition of star-K ($\bigstar$-$\mathcal{K}$) transformations, a process we previously called *interiorizing.* These ideas go back to David Ingerman in 1991. The Schur complement is just block Gaussian elimination and as such can be done one step at a time. This is the $\bigstar$-$\mathcal{K}$ transformation. If each composition can be reversed the problem is solved. Even if there is some ambiguity which can eventually be eliminated the problem is solved. This is a very complex process that seems to generally be more complicated than other methods, but it applies to any graph, not just circular planar graphs. Nick Addington wrote a program to do this. His work is posted under the papers for the 2003 REU program, but was not actually completed until 2005. It can use some revision. It has proved useful, but it is not user-friendly.

This gets back to the Grobner basis idea. Is there some method that doesn't required cleverness or geometrical insight? I need to "look" at the graph to solve most problems. There is a routine that produces the medial graph without "looking" at a critical circular planar graph. It works by computing ranks of submatrices of Λ and using the cut point lemma, $m = n - r$ to produce the $Y - \Delta$ equivalence class (the *z-sequence*) of the medial graph. But we still need to know the graph itself to compute conductivities.

6. DIRECTED GRAPHS; GRAPHS WITH COMPLEX CONDUCTIVITIES OR CONDUCTIVITIES TAKEN FROM A FIELD. Some results by Michael Goff, Sam Coskey, Mark Blunk, Orion Bawdon, Joel Nishimura, and Peter Mannisto. Even the forward problem needs thought. What is the substitute for positive semi-definiteness if we decide to use fields? Is there a substitute for the alternating principle and signs of determinants?