An Introduction to Paramodular Forms

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Modular Forms

The group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\pm 1$ acts on the complex upper half-plane

$$\mathfrak{H} := \{z \in \mathbb{C} : \mathcal{I}(z) > 0\}$$

by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Functions on $\mathfrak{H}/\mathrm{PSL}_2(\mathbb{Z})$ are called *modular functions*

Modular Forms

A *modular form*, is a holomorpic function on $\mathfrak{H}$ which satisfies growth conditions and for every

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \subset \mathrm{PSL}_2(\mathbb{Z}),$$

$$f|_\gamma(z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

Note that $\gamma \in \Gamma_0(N)$ if $c \equiv 0 \bmod (N)$. We denote the space of modular forms of weight k and level N by $M_k(N)$.

Fourier coefficients

Since $f(z+1) = f(z)$, we have a periodic function and can consider the Fourier expansion

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z} = \sum_{n \geq 0} a_n q^n$$

Note that as $z \to i\infty$, $q \to 0$ so one can view this as a Taylor series expansion about $q = 0$ or the cusp $z = i\infty$. The growth conditions on f guarantee the Fourier expansion about the other cusps of $\Gamma_0(N)$ as well.

Newforms

- If $a_0 = 0$ for the Fourier expansion of f about *every* cusp of $\Gamma_0(N)$, then we way that f is a *cusp form*. We denote the space of cusp forms by $S_k(N)$.
- Notice that if $n \mid N$, then we have that $S_k(n) \subset S_k(N)$. Moreover, one can make $S_k(N)$ into a Hilbert space via the Peterssen inner product.
- Let S_k^{old} be the subspace spanned by $S_k(n)$. Then S_k^{new} is its orthogonal complement

Hecke Operators

- For $m \in \mathbb{N}$ and $f \in M_k(N)$, Hecke defined

$$(T_m f)(z) = \sum_{n \geq 0} \left(\sum_{d | \gcd(m,n)} d^{k-1} a_{mn/d^2} \right) q^n$$

- T_m and T_n commute and $S_k(N)$ has a basis of Hecke eigenforms.
- If we normalize f so that $a_1 = 1$, then $T_m f = a_m f$
- If $\gcd(m, n) = 1$, $a_m a_n = a_{mn}$!

L-functions

- The L-function of f is defined simply via the Dirichlet series

$$L(s,f)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

- $L(s,f)$ converges for $Re(s)>\frac{k}{2}+1$, has analytic continuation, an Euler product expansion, and satisfies a functional equation.

Elliptic Curves and the Modularity Theorem

Theorem *If E is an elliptic curve over $\mathbb{Q}$ of conductor N, then there is a modular form f of weight 2 and conductor N such that $L(s, F) = L(s, E)$.*

Automorphic Forms

- Identify $\mathfrak{H}$ with $\mathrm{PSL}_2(\mathbb{R})/(\mathrm{SO}(2)/\pm 1)$ and lift f to $\mathrm{PSL}_2(\mathbb{R})$.
- Define the automorphic form attached to f via.

$$\phi_f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (cz+d)^k f \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- For $\gamma \in \Gamma$, $\phi_f(\gamma g) = \phi_f(g)$.
- For $k \in \mathrm{SO}(2)/\pm 1$, $\phi_f(gk) = \pi(k)\phi_f(g)$ and the vector space $<\phi_f(gk)> \subset C^\infty(\mathrm{PSL}_2(\mathbb{R}))$ is finite dimensional.

Automorphic Forms

- The conditions that a modular form be holomorphic and well-behaved at the cusps is also translated into this language.
- Using parabolic subgroups, we can define cuspidal automorphic forms.
- Further we an adelic formulation using

$$\Gamma_0(N) \setminus \mathrm{PSL}_2(\mathbb{R}) \simeq Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A})/K_N$$

where $G = \mathrm{GL}(2), Z = \mathrm{GL}(1)$ and $K_N \subset \prod_{p<\infty} G(\mathbb{Z}_p) \times \mathrm{SO}(2)$.

Automorphic Representations

- To construct an automorphic representation, we consider the space of L^2 automorphic forms. Then G acts on this space by right translation

$$(g \cdot \phi)(x) = \phi(xg)$$

- The space of cusp forms decomposes under this action into a sum of irreducible representations which are called the *cuspidal automorphic representations* of G. We denote an automorphic representation by (π, V)
- $\pi = \otimes_v \pi_v$ where each π_v is a representation of $G(\mathbb{Q}_p)$
- We obtain an automorphic representation $(\pi_f, G(\mathbb{A}) \cdot \phi_f)$ and the local representations are completely determined by the Hecke operators and the weight of f!

Local Newforms

Theorem *If (π, V) is a "nice" representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ there there is some n for which the space of vectors invariant under $\Gamma_0(p^n)$, denoted $V(n)$ is nonzero. Moreover, if N_π is the least such integer, then the dimension of $V(N_\pi)$ is 1. Finally, every element of V can be obtained from $V(N_\pi)$ by applying level-raising operators and taking linear combinations.*

This theorem is very powerful as it enables the computation of epsilon factors, Hecke eigenvalues and the the L factors.

GSp(4)

- Let
$$J = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$$
- Define the algebraic group GSp(4) to be the set of all $g \in \mathrm{GL}(4)$ such that ${}^t g J g = \lambda(g) J$ for some $\lambda \in \mathrm{GL}(1)$.

Siegel Modular Forms

The group $G\mathrm{Sp}(4,\mathbb{R})+$ (where $\lambda(g) > 0$ acts on the Siegel upper half-plane

$$\mathfrak{H}_2 := \{\mathbb{Z} \in \mathrm{Mat}(2 \times 2, \mathbb{C}) : \mathcal{I}(\mathrm{Z}) > 0\}$$

by transformations

$$g < Z >= (AZ + b)(CZ + D)^{-1}$$

where $g = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$.

Siegel Modular Forms

A *Siegel modular form*, is a holomorpic function on $\mathfrak{H}_2$ such that for every

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_(4, \mathbb{Z}),$$

$$F|_g(z) = \lambda(g)^k \det(CZ + D)^{-k} f F(g < Z >) = F(Z).$$

These are Siegel modular forms of paramodular level one, but they are also of level one for the Siegel congruence subgroups. The theory is quite varied from here, and we will focus on the paramodular groups.

Paramodular Group

For N a positive integer we define

$$\Gamma^{\text{para}}(N) = \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}(4, \mathbb{Q}).$$

Notice that $\Gamma^{\text{para}}(N) \not\subset \Gamma^{\text{para}}(Np)$.

Paramodular Forms

A *Paramodular form*, is a holomorpic function on $\mathfrak{H}_2$ such that for every

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{\text{para}}(N),$$

$$F|_g(Z) = \lambda(g)^k \det(CZ + D)^{-k} F(g < Z >) = F(Z).$$

Paramodular Automorphic Representations

- Given cuspidal, irreducible, admissible representation π of $\mathrm{GSp}(4, \mathbb{A})$ which is invariant under Γ^{para}
- We construct a Siegel modular form by choosing a vector Φ.
- Define $F : \mathfrak{H}_2 \to \mathbb{C}$ by $F(Z) = \lambda(h)^{-k} j(h, I)^k \Phi(h_\infty)$ where $h \in \mathrm{GSp}(4, \mathbb{R})^+$ is such that $h\langle I \rangle = Z$.

Old and New Forms

Theorem *(Roberts, Schmidt) If (π, V) is a "nice" representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ there there is some n for which the space of vectors invariant under $\Gamma^{\mathrm{para}}(p^n)$, denoted $V(n)$ is nonzero.*
Moreover, if N_π is the least such integer, then the dimension of $V(N_\pi)$ is 1.
Finally, every element of V can be obtained from $V(N_\pi)$ by applying level-raising operators and taking linear combinations.

This theorem is very powerful as it enables the computation of epsilon factors, Hecke eigenvalues and the L factors.

The Paramodular Conjecture

Conjecture *(Brumer and Kramer) There is a one-to-one correspondence between isogeny classes of abelian surfaces A over $\mathbb{Q}$ of conductor N with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and weight 2 newforms F on $\Gamma^{\mathrm{para}}(N)$) with rational eigenvalues, not in the span of the Gritsenko lifts, such that $L(A, s) = L(F, s)$. The ℓ-adic representations associated to F should be isomorphic to those of the Tate module of A for any ℓ prime to N.*

Hecke Operators

We have two Hecke operators $T(1,1,p,p)$ and $T(1,p,p,p^2)$ on $M_k(\Gamma^{\text{para}}(N))$ as follows. Let

$$\Gamma^{\text{para}}(N)\begin{bmatrix}1&&&\\&1&&\\&&p&\\&&&p\end{bmatrix}\Gamma^{\text{para}}(N)=\bigsqcup\Gamma^{\text{para}}(N)h_i$$

and

$$\Gamma^{\text{para}}(N)\begin{bmatrix}p&&&\\&1&&\\&&p&\\&&&p^2\end{bmatrix}\Gamma^{\text{para}}(N)=\bigsqcup\Gamma^{\text{para}}(N)h_j'$$

A Theorem

- Let E be a real quadratic extension of $\mathbb{Q}$ with real archimedean places ∞_1 and ∞_2

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A Theorem

- Let E be a real quadratic extension of $\mathbb{Q}$ with real archimedean places ∞_1 and ∞_2
- Let f be a Hilbert modular form for E of weight (2, 2n+2) and level $\mathfrak{N}_0$.
- To f we attach π_0, a cuspidal irreducible automorphic representation of $\mathrm{GL}(2, \mathbb{A}_E)$ with trivial central character.

Statement of Theorem

Assume that π_0 is not Galois-invariant and $\pi_{0,\infty_1} = D_2$ and $\pi_{0,\infty_2} = D_{2n+2}$ with $n \geq 0$ a non-negative integer and D_k the holomorphic discrete series representation of weight k. Let $N = d_E^2 \mathrm{N}_{\mathbb{Q}}^{E}(\mathfrak{N}_0)$, where d_E is the discriminant of $E/\mathbb{Q}$.
Then there exists a non-zero Siegel paramodular newform $F : \mathfrak{H}_2 \to \mathbb{C}$ of weight $k = n + 2$ and paramodular level N. We describe the properties of F below.

Hecke Eigenvalues

For every prime p, let $T(1,1,p,p)$ and $T(1,p,p,p^2)$ be the Hecke operators on the space of Siegel modular forms of degree two. Then

$$T(1,1,p,p)F = p^{k-3}\lambda_p F \quad \text{and} \quad T(1,p,p,p^2)F = p^{2(k-3}$$

where the Hecke eigenvalues λ_p and μ_p are determined by the Hecke eigenvalues of π_0 as follows. If p splits, let w_1 and w_2 be the places above p. If p is non-split, let w be the place above p.

- If $\mathrm{val}_p(N) = 0$,

$$\lambda_p = \begin{cases} p(\lambda_{w_1} + \lambda_{w_2}) & \text{if } p \text{ is split,} \\ 0 & \text{if } p \text{ is not split,} \end{cases}$$

$$\mu_p = \begin{cases} p^2 + p\lambda_{w_1}\lambda_{w_2} - 1 & \text{if } p \text{ is split,} \\ -(p^2 + p\lambda_w + 1) & \text{if } p \text{ is not split.} \end{cases}$$

- If $\mathrm{val}_p(N) = 1$, then p splits and $\mathrm{val}_{w_1}(\mathfrak{N}_0) = 1$, $\mathrm{val}_{w_2}(\mathfrak{N}_0) = 0$, and

$$\lambda_p = p\lambda_{w_1} + (p+1)\lambda_{w_2}, \qquad \mu_p = p\lambda_{w_1}\lambda_{w_2}.$$

If $\mathrm{val}_p(N) \geq 2$, then:

- p inert:

$$\lambda_p = 0, \qquad \mu_p = -p^2 - p\lambda_w;$$

- p ramified:

$$\lambda_p = p\lambda_w, \qquad \mu_p = \begin{cases} 0 & \text{if } \mathrm{val}_w(\mathfrak{N}_0) = 0, \\ -p^2 & \text{if } \mathrm{val}_w(\mathfrak{N}_0) \geq 1; \end{cases}$$

- p split and $\mathrm{val}_{w_1}(\mathfrak{N}_0) \leq \mathrm{val}_{w_2}(\mathfrak{N}_0)$:

$$\lambda_p = p(\lambda_{w_1} + \lambda_{w_2}), \qquad \mu_p = \begin{cases} 0 & \text{if } \mathrm{val}_{w_1}(\mathfrak{N}_0) = 0, \\ -p^2 & \text{if } \mathrm{val}_{w_1}(\mathfrak{N}_0) \geq 1. \end{cases}$$

Atkin-Lehner Eigenvalues

For every prime $p|N$, let U_p be the Atkin-Lehner operator. Then,

$$F|_k U_p = \varepsilon_p F$$

with

$$\varepsilon_p = \begin{cases} \varepsilon(1/2, \pi_{0,w_1}, \psi_p, dx_{\psi_p})\varepsilon(1/2, \pi_{0,w_2}, \psi_p, dx_{\psi_p}) & \text{spli} \\ \varepsilon(1/2, \pi_{0,w}, \psi_w, dx_{\psi_w})\omega_{E_w/\mathbb{Q}_p}(-1) & \text{not} \end{cases}$$

where ψ_w is an additive character of E_w with conductor $\mathfrak{o}_w$.

L-function

For every prime p, we have an equality of Euler factors

$$L_p(s+k-\frac{3}{2},F)=L_p(s,\pi_0),$$

where $k=n+2$ and $L_p(s,F)$ is defined below for every finite place p of $\mathbb{Q}$. Moreover, we have the functional equation

$$\Lambda(2k-2-s,F)=(-1)^k(\prod_{p|N}\varepsilon_p)N^{s-k+1}\Lambda(s,F)$$

where the completed L-function is defined as the product

$$\Lambda(s,F)=(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)\prod_{p<\infty}L_p(s,F).$$