① **Dokchitser's Algorithm** | Stein

General Algorithm to compute $L^{(r)}(s)$ "in seconds" for any $r \geq 0$, $s \in \mathbb{C}$ and any motivic L-functions.

- Completely general
- Fast enough for some apps: e.g. near real axis, very good for me — not for Mike R.

Implementations:

- gp-pari ~ 500 "lines" of code — Demo live
- magma: ~ 1500 "lines"

Pari version is 2–5 × faster in my benchmarks than magma.

Applications:
- BSD, Bloch-Kato, Stark conjectures, (Gross-Zagier + Zhang) Heegner points ($L_A(f,s)$, $L(f,\chi,s)$)
- computing conductor, bad factors
- Computing Petersson pairings, ε, etc.
- Poincare series
- modular degrees ($\sim L(\mathrm{Sym}^2(f), 2)$)

② **Setup:**

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad a_n \in \mathbb{C}, \text{ polynomial growth}$$

$$\gamma(s) = \Gamma\left(\frac{s+\lambda_1}{2}\right) \cdots \Gamma\left(\frac{s+\lambda_d}{2}\right) \quad \text{'Hodge numbers' } \lambda_i \in \mathbb{C}$$

$$L^*(s) = A^s \gamma(s) L(s) \quad \text{exponential factor } A \text{ (conductor-ish)} = \frac{\sqrt{N}}{\pi^{d/2}}$$

Functional equation: $L^*(s) = \varepsilon L^*(w - s)$, $\varepsilon \in \mathbb{C}$ 'sign', w 'weight' $\in \mathbb{Z}_{\geq 0}$

Finitely many poles p_j w/ residues $r_j = \mathrm{res}_{s=p_j} L^*(s)$.

INPUT: $a_n, \lambda_i, A, \varepsilon, w, p_j, r_j$.

OUTPUT: Formula to approx. $L^{(k)}(s)$ $\forall k, s$.

③ **Strategy**

$$L(s) = \frac{L^*(s)}{A^s \gamma(s)},$$ so we focus on $L^*(s)$ instead.

Theorem: $\exists$ function $\theta(t): \mathbb{R}_{>0} \to \mathbb{C}$ such that

$$\boxed{L^*(s) = \int_0^\infty \theta(t) t^s \frac{dt}{t}}$$

$\theta(t)$ "inverse Mellin transform"

" $L^*(s)$ is the Mellin transform of $\theta(t)$ "

"Proof": In fact, $\theta(t) = \int_{c-i\infty}^{c+i\infty} L^*(s) t^{-s} ds$ any c with $\mathrm{Re}(c) \gg 0$

Mellin inversion formula. (no zeros to right of c) ∎

④ Given $\theta(t)$ can compute $L^*(s)$:

Formula for $L^*(s)$:

$$L^*(s) = \int_0^\infty \theta(t) t^s \frac{dt}{t}$$

③ $= \cdots$ calculus $\cdots$ ($\cdots$ series/integral manip) LATER

$$= \sum_{n=1}^{\infty} a_n G_s\left(\frac{n}{A}\right) + \varepsilon \sum_{n=1}^{\infty} a_n G_{w-s}\left(\frac{n}{A}\right) + \sum \frac{r_j}{p_j - s}$$

poles

② Let $G_s(t) \overset{\text{def}}{=} t^{-s} \int_t^\infty \phi(x) x^s \frac{dx}{x}$ and

① $\phi(t)$: $\gamma(s) = \int_0^\infty \phi(t) t^s \frac{dt}{t}$ "inverse Mellin transform of $\gamma(s)$".

gen. γ factor — depends only on γ!

⑤ Also get $\frac{\partial^k}{\partial s^k} L^*(s)$ by trivial term-by-term differentiation.

[So... we "just" need to compute this function $\boxed{\frac{\partial^k}{\partial s^k} G_s(t)}$ which depends $\boxed{\text{only on } \gamma(s)}$.] !

⑥ Computing $\boxed{G_s(t)}$ (function of $\gamma(s)$ only)

Prop:

$$G_s(t) = \frac{\gamma(s) - \int_0^t \phi(x) x^s \frac{dx}{x}}{t^s}$$

(when s not pole of $\gamma(s)$ (or t^s))

Proof

$$t^s G_s(t) = t^s \left(t^{-s} \int_t^\infty \phi(x) x^s \frac{dx}{x}\right)$$

$$= \int_t^\infty \phi(x) x^s \frac{dx}{x} = \gamma(s) - \gamma(s) + \int_t^\infty \phi(x) x^s \frac{dx}{x}$$

$$= \gamma(s) - \int_0^\infty \phi(x) x^s \frac{dx}{x} + \int_t^\infty \phi(x) x^s \frac{dx}{x}$$

$$= \gamma(s) - \int_0^t \phi(x) x^s \frac{dx}{x}. \quad \blacksquare$$

Now just need explicit series for $\phi(x)$.

⑦ Prop (Mellin inversion formula)

$$\phi(x) \; (= \text{inverse Mellin transform of } \gamma(s))$$
$$= \sum_{\substack{\text{poles} \\ z \text{ of } \gamma}} \operatorname{res}_{s=z} \underbrace{(\gamma(s) x^{-s})}_{\text{function of } s}$$

⇝ explicit power series in x with coeff. poly's in $\log(x)$.

⇝ Can compute $G_s(t)$ by integrating term-by-term using above prop. Likewise for $\frac{\partial^k}{\partial x^k} G_s(t)$.

DONE

⑧ Return to Formula for $L^*(s)$.

Prop:
$$L^*(s) = \sum_{n=1}^{\infty} a_n G_s\left(\frac{n}{A}\right) + \varepsilon \sum_{n=1}^{\infty} a_n G_{w-s}\left(\frac{n}{A}\right) + \sum_j \frac{r_j}{p_j - s}$$

Proof:
$$L^*(s) = \int_0^\infty \theta(t) t^s \frac{dt}{t} \qquad \boxed{\text{by defn of } \theta(t)}$$
$$= \int_1^\infty \theta(t) t^s \frac{dt}{t} + \int_0^1 \theta(t) t^s \frac{dt}{t} \qquad \boxed{\text{trivial}}$$
$$= \int_1^\infty \theta(t) t^s \frac{dt}{t} + \int_1^\infty \theta\left(\frac{1}{t}\right) t^{-s} \frac{dt}{t}$$

Change of variables: $y = 1/t$, $t = 1/y$, $\frac{dt}{t} = \frac{d(1/y)}{1/y} = -y \cdot \frac{1}{y^2} dy$; $-\int_\infty^1 \theta(1/y) y^{-s} \frac{dy}{y}$

$$= \int_1^\infty \theta(t) t^s \frac{dt}{t} + \int_1^\infty \varepsilon t^w \theta(t) t^{-s} \frac{dt}{t} - \int_1^\infty \sum_j r_j t^{p_j} t^{-s} \frac{dt}{t}$$

Uses functional equation
$$\theta(1/t) = \varepsilon t^w \theta(t) - \sum_j r_j t^{p_j}$$
$$\theta(1/t) = \int_{c-i\infty}^{c+i\infty} L^*(s) t^s ds$$
$$= t^w \int_{c-i\infty}^{c+i\infty} \varepsilon L^*(w-s) t^{s-w} ds$$
$$= t^w \varepsilon \underbrace{\int_{w-c-i\infty}^{w-c+i\infty} L^*(s) t^{-s} ds}_{\text{almost } \theta(1/t)}$$

change vars $y = w-s$ & switch direction of $\int$.

but pick up poles since $w-c+i\infty$ is to left of $\mathrm{Re}(s)=c$.
$$= \varepsilon t^w \theta(t) - \sum_j r_j t^{p_j}$$

$$= \int_1^\infty \theta(t) t^s \frac{dt}{t} + \varepsilon \int_1^\infty \theta(t) t^{w-s} \frac{dt}{t} + \underbrace{\sum_j \frac{r_j}{p_j - s}}_{\text{easy calculus}}$$

same argument

$$\int_1^\infty \sum_{n=1}^\infty a_n \phi\left(\frac{nt}{A}\right) t^s \frac{dt}{t}$$

Prop: $\theta(t) = \sum_{n=1}^\infty a_n \phi\left(\frac{nt}{A}\right)$, where θ is the inv Mellin of L^* and ϕ is the inv Mellin of $\gamma(s)$.

Proof: By defn
$$L^*(s) = \int_0^\infty \theta(t) t^s \frac{dt}{t} \quad \text{characterizes } \theta(t).$$
But
$$\int_0^\infty \left(\sum_{n=1}^\infty a_n \phi\left(\frac{nt}{A}\right)\right) t^s \frac{dt}{t} = \sum_{n=1}^\infty a_n \int_0^\infty \phi\left(\frac{nt}{A}\right) t^s \frac{dt}{t}$$
$$= \sum_{n=1}^\infty a_n \int_0^\infty \phi(t) \left(\frac{At}{n}\right)^s \frac{dt}{t} \quad \leftarrow \boxed{\text{change of var } y = \frac{n}{A} t, \; dt = \frac{A}{n} dy}$$
$$= \left(A^s \sum_{n=1}^\infty \frac{a_n}{n^s}\right) \gamma(s) = L^*(s). \Rightarrow \text{the prop.}$$

$$= \sum_{n=1}^\infty a_n \int_1^\infty \phi\left(\frac{nt}{A}\right) t^s \frac{dt}{t} \underset{\substack{\text{change of var} \\ y = \frac{nt}{A}}}{=} \sum_{n=1}^\infty a_n \underbrace{\int_{n/A}^\infty \phi(t) \left(\frac{At}{n}\right)^s \frac{dt}{t}} = \sum_{n=1}^\infty a_n G_s\left(\frac{n}{A}\right). \blacksquare$$