Toric Geometry in Sage and String/F-Theory Applications

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Toric Geometry

Definition (Toric Variety)

A *toric variety* is a variety with an algebraic $T = (\mathbb{C}^\times)^n$ action such that there is a single n-dimensional orbit $\simeq T$.

Example

$\mathbb{P}^2$ with the $(\mathbb{C}^\times)^2$-action

$$(\mathbb{C}^\times)^2 \times \mathbb{P}^2 \to \mathbb{P}^2, \quad (\mu, \nu)\big([z_0 : z_1 : z_2]\big) = [z_0 : \mu z_1 : \nu z_2]$$

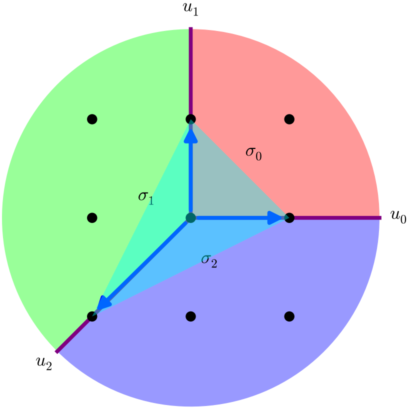
is a toric variety.

In the example, the top-dimensional orbit is $[1 : \mu : \nu]$.

Definition (Fan)

A *fan* is a collection of cones in a lattice $\simeq \mathbb{Z}^n$ such that

- Every face of a cone is another cone of the fan.
- Any two cones of a fan intersect in a common face.



7 cones:

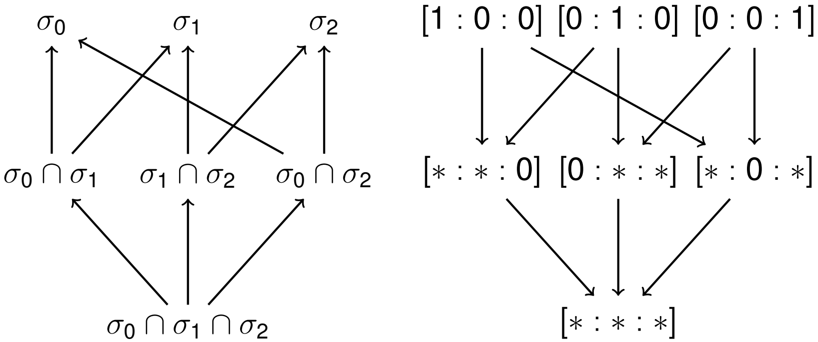
$\sigma_0, \sigma_1, \sigma_2,$

$\sigma_0 \cap \sigma_1, \sigma_0 \cap \sigma_2, \sigma_1 \cap \sigma_2$

$\{0\} = \sigma_0 \cap \sigma_1 \cap \sigma_2$

$\nabla = \mathrm{conv}(\text{rays})$

Poset of Cones vs. Poset of Torus Orbits



d-dimensional cone $\quad\leftrightarrow\quad$ $n - d$-dimensional torus orbit

Toric Geometry Wishlist

Want an efficient implementation of toric algorithms!

- Integers & rational numbers `gmp, MPIR`
- Lattices (Smith normal form, LLL) `ntl`
- Convex geometry (linear programs, dual description) `cddlib, PPL, glpk, Polymake`
- Lattices and polytopes (mixed integer programming) `palp, PPL`
- Triangulations `TOPCOM`
- Graphs (automorphisms) `nauty`
- Polynomials (Groebner bases) `Singular, M2`

State of the Union

Toric geometry implementations suck!

- Bits & pieces implemented by different programs.
- No common framework.
- Reinventing the wheel.
- Either focus too narrow or too slow.

C/C++ vs. interpreter.

- Unit tests?
- Code review?
- Bug tracker?
- Documentation?
- Revision control system?

Sage (http://www.sagemath.org)

Mission: Creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab. [W. Stein et al.]

- Python — Easy to write & easy to read!
- Cython — Combine native C/C++ with Python.
- Incorporates many existing open-source packages.
- Batteries included.
  All necessary libraries, documentation, ...
- Software engineering best practices:
  - Code must have documentation.
  - Code must be peer reviewed.
  - Code must have an automated test.
  - Automated testing on a variety of architectures.

Toric Geometry in Sage

Goal: Implement all toric algorithms.

[VB and A. Novoseltsev]

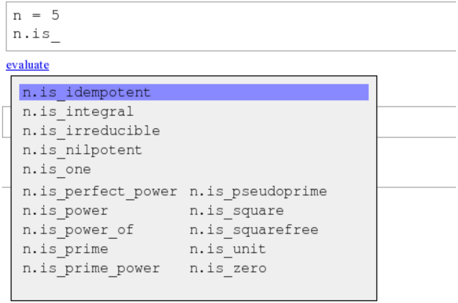
- Polytopes
  Dual description, integral points, triangulations, graphics, ...
- Cones
  Convex geometry, Hilbert basis, lattice constructions, ...
- Fans
  Cone lattice, Stanley Reisner ideal, ...
- Toric varieties
  Cohomology, Chow group, Kähler/Mori cone,
- Toric divisors / sheaves
  Sheaf cohomology, Weil vs. Cartier, Chern classes, index theory, ...

Object-Oriented Programming

Data comes first!

```
sage: n = 5
sage: is_prime(n)      # procedural
True
sage: n.is_prime()     # object-oriented
True
```

Use tab completion to explore methods:



The Chow Group

$$A_k(X) = \Big\{\text{algebraic } k\text{-cycles}\Big\} \Big/ \Big\{\text{rational equivalence}\Big\}$$

- For example, $A_{n-1}(X)$ = (Weil) divisor class group.
- If the toric variety X is smooth, then $A_k(X) \simeq H^{2k}(X, \mathbb{Z})$.
- Works for singular toric varieties!

Engineering Singularities

Let's construct a Calabi-Yau variety with singularities.
Idea:

- Start with 4-d toric variety with curve C of singularities.
- Calabi-Yau hypersurface D intersects C in some number $C \cdot D$ of points.

A Singular Toric Variety

The weighted projective space $\mathbb{P}^4_{11133}$ has a curve of $\mathbb{Z}_3$ orbifold singularities:

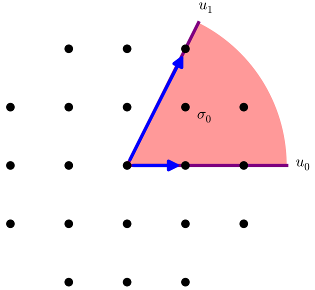
```
sage: v = matrix([[ 1, 0, 0, 0],[ 0, 1, 0, 0],
...               [ 0, 0, 1, 0],[ 0, 0, 0, 1],
...               [-1,-1,-3,-3]])
sage: Nabla = LatticePolytope(v.transpose())
sage: fan = FaceFan(Nabla)
sage: [ cone.rays() for cone in fan(3)
...     if not cone.is_smooth() ]
[(N(1, 0, 0, 0), N(0, 1, 0, 0), N(-1, -1, -3, -3))]
```

Definition (Smooth cone)

- the rays generate all integral points.
- Equivalently, the torus orbit contains no singularities.

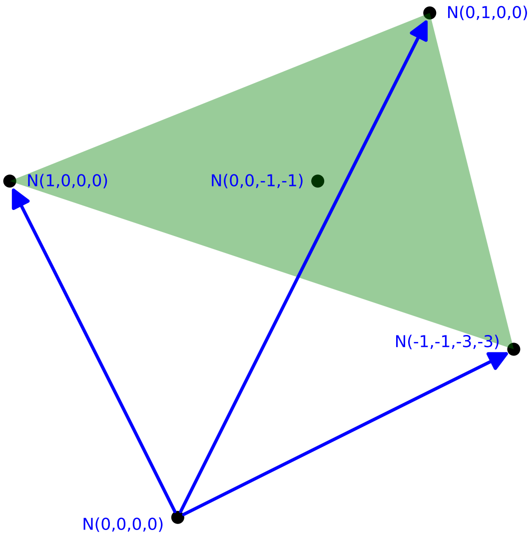
The Hilbert Basis of a Cone

Standard example: $\mathbb{C}^2/\mathbb{Z}_2 \quad =$



The singular 3-cone of the 4-d fan:

```
sage: C = Cone([(1,0,0,0),(0,1,0,0),(-1,-1,-3,-3)])
sage: C.Hilbert_basis()
(N(-1, -1, -3, -3), N(0, 1, 0, 0), N(1, 0, 0, 0), N
    (0, 0, -1, -1))
```



This cone alone defines a $\mathbb{C}^3/\mathbb{Z}_3$ singularity.

$\Rightarrow$ A curve $C \simeq \mathbb{P}^1$ of $\mathbb{C}^3/\mathbb{Z}_3$ singularities in the 4-d toric variety.

Naive Intersection Computation

How often does C intersect the Calabi-Yau hypersurface D?

Use description of $H^\bullet(X)$ as quotient by Stanley-Reisner ideal:

$$H^\bullet(X, \mathbb{Q}) \simeq \mathbb{Q}[z_0, \ldots, z_d]/(SR + L)$$

where

- SR is the Stanley-Reisner ideal, and
- L is the ideal of linear equivalences.

Naive Intersection Computation cont'd

```
sage: P = ToricVariety(fan)  # P^{11133}
sage: P.cohomology_basis()
(([1],), ([z4],), ([z4^2],), ([z4^3],), ([z4^4],))
sage: D = -P.K()  # anticanonical divisor
sage: H = P.cohomology_ring()
sage: H(D) * H(C)
[9*z4^4]
sage: P.integrate( H(D) * H(C) )
1
```

Attention

Wrong! A variety with orbifold singularities does not have (integer) Poincaré duality!

Intersections in the Chow Group

In fact, the correct intersection number is $C \cdot D = 3$.

The correct intersection number can be computed using Chow homology, without resorting to Poincaré duality:

```
sage: P = ToricVariety(fan)  # P^{11133}
sage: D = -P.K()  # anticanonical divisor
sage: A = P.Chow_group()
sage: A.degree()
(Z, Z, Z, Z, Z)
sage: A(D)
( 0 | 0 | 0 | 9 | 0 )
sage: A(C)
( 0 | 1 | 0 | 0 | 0 )
sage: A(C).intersect_with_divisor(D)
( 3 | 0 | 0 | 0 | 0 )
sage: A(C).intersect_with_divisor(D).count_points()
3
```

F-Theory

F-theory compactification requires an elliptic fibration with

- X a 4-d Calabi-Yau variety, and
- B a 3-d variety.

$$\begin{array}{ccc} T^2 & \longrightarrow & X \\ & & \Big\downarrow \pi \\ & & B \end{array}$$

The discriminant

$$\Delta = \bigcup_i D_i \quad \subset B$$

is where the fiber differs from T^2.

Kodaira type of discriminant = (local) gauge group.

Surfaces with Fundamental Groups

The actual low energy gauge group depends on more (global) data.

- “split” vs. “non-split” discriminant.
- Discrete Wilson lines if $\pi_1(D_i) \neq 1$.
- For example: Enriques surface = $K3/\mathbb{Z}_2$.

Naive idea: Take quartic in $B = \mathbb{P}^3_{[x_0:x_1:x_2]}$ and divide out $\mathbb{Z}_2$, acting freely on the quartic. Does not work!

Enriques surfaces are not free quotients of quartics.

An Enriques Surface

Let S be a quartic invariant under the $\mathbb{Z}_4$ action

$$g : \mathbb{P}^3 \to \mathbb{P}^3, \quad \big[x_0 : x_1 : x_2 : x_3\big] \mapsto \big[x_0 : ix_1 : i^2 x_2 : i^3 x_3\big]$$

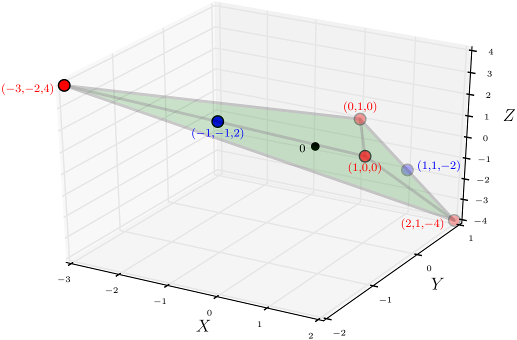
Projects out the $(0,2)$-form on the quartic:

$$\Omega^{(2,0)} = \oint \frac{\epsilon^{ijk\ell} x_i \, \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_\ell}{q(x_0, x_1, x_2, x_3)}$$

- Curves of $\mathbb{Z}_2$ fixed points in $\mathbb{P}^3$.
- $S/\mathbb{Z}_4$ is a singular Enriques surface

Toric Version

$B = \mathbb{P}^3 / \mathbb{Z}_4$ as a toric variety.



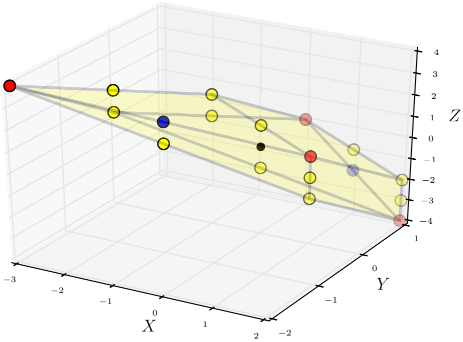
- blue vertices = curves of $\mathbb{Z}_2$ singularities
- $A_2(B) = \mathbb{Z} \times \mathbb{Z}_4$

Cox homogeneous coordinates construction:

$$B = \frac{\mathbb{C}^4 - \{0\}}{\mathrm{Hom}\left(A_2(X), \mathbb{C}^\times\right)} = \frac{\mathbb{C}^4 - \{0\}}{\mathbb{C}^\times \times \mathbb{Z}_4}$$

Toric Version

Resolution $\hat{B}$ of $\mathbb{P}^3/\mathbb{Z}_4$.



- Not crepant.
- Reflexive polytope.
- Proper transform $\hat{D}$ of singular Enriques D is smooth.
- Origin + 18 points.

Divisors on the Resolution

Homogeneous coordinates $\hat{z}_0, \ldots, \hat{z}_{17}$ of $\hat{B}$.

$$
\begin{aligned}
-K_{\hat{B}} &= \sum_{i=0}^{17} V(\hat{z}_i) \\
\hat{D} &= 4V(\hat{z}_0) + 2V(\hat{z}_4) + 4V(\hat{z}_6) + 3V(\hat{z}_7) + \\
&\quad 3V(\hat{z}_8) + 2V(\hat{z}_9) + 2V(\hat{z}_{10}) + 2V(\hat{z}_{11}) + \\
&\quad V(\hat{z}_{12}) + 2V(\hat{z}_{13}) + V(\hat{z}_{14})
\end{aligned}
$$

- Kähler cone is 15-d cone spanned by 169 rays.
- $-K_{\hat{B}}$, $\hat{D}$ are on different faces of the Kähler cone.

Weierstrass Model

Every elliptic fibration Y has a Weierstrass model

$$W = \left\{ y^2 z = x^3 + a_1 xyz + a_2 x^2 z^2 + a_3 yz^3 + a_4 xz^4 + a_6 z^6 \right\}$$
$$\subset \mathbb{P}\Big(\mathcal{O} + \mathcal{O}(-2K) + \mathcal{O}(-3K) \Big)$$

"model" in the sense of the MMP:
W is birational to Y and "simpler".

Since we don't have a particular elliptic fibration in mind:
Just construct Weierstrass model on $\hat{B}$.

Weierstrass Model cont'd

$$y^2 z = x^3 + a_1 xyz + a_2 x^2 z^2 + a_3 yz^3 + a_4 xz^4 + a_6 z^6$$

To make sense globally, must chose

$$x \in \Gamma\mathcal{O}(-2K),\ y \in \Gamma\mathcal{O}(-3K),\ z \in \Gamma\mathcal{O} \quad \Rightarrow \quad a_\ell \in \Gamma\mathcal{O}(-\ell K)$$

Gauge group depends on degree of vanishing of a_ℓ along the Enriques surface $\hat{D}$.

$$\left\{ \begin{array}{l} \text{There exists a suitable} \\ \text{section for } a_\ell \text{ vanishing} \\ \text{to degree } \delta \text{ along } \hat{D} \end{array} \right\} \Leftrightarrow H^0\big(-\ell K - \delta \hat{D}\big) \neq 0$$

The Smooth Enriques Surface in Sage

```
sage: rays = [(-3,-2,4), (0,1,0), (1,0,0), (2,1,-4),
     (-1,-1,2), (1,1,-2), (-2,-1,2), (-2,-1,3),
    (-1,-1,1), (-1,0,1), (-1,0,2), (0,0,-1), (0,0,1)
    , (1,0,-2), (1,0,-1), (1,1,-1), (2,1,-3),
    (2,1,-2)]
sage: cones = [(0,3,11), (0,6,11), (5,6,11),
    (3,5,11), (0,3,8), (3,13,8), (4,13,8), (0,4,8),
    (2,3,16), (2,17,16), (3,5,16), (5,17,16),
    (4,13,14), (2,4,14), (2,3,14), (3,13,14),
    (5,6,9), (1,5,9), (0,1,9), (0,6,9), (0,4,7),
    (4,10,7), (0,1,7), (1,10,7), (1,5,15), (5,17,15)
    , (1,2,15), (2,17,15), (4,10,12), (2,4,12),
    (1,10,12), (1,2,12)]
sage: Bsmooth = ToricVariety(Fan(cones, rays))
sage: K = Bsmooth.K()
sage: Dsmooth = Bsmooth.divisor([4, 0, 0, 0, 2, 0,
    4, 3, 3, 2, 2, 2, 1, 2, 1, 0, 0, 0])
```

Sheaf Cohomology in Sage

For example, the sheaf cohomology groups of $\mathcal{O}(3\hat{D} + 4K_{\hat{B}})$ are

```
sage: (3*Dsmooth+4*K).cohomology(dim=True)
(0, 0, 24, 0)
```

Compute a matrix whose entries are the relevant cohomology groups:

```
sage: matrix(ZZ, 5, 7, lambda d,l:
...      len((-l*K-d*Dsmooth).sections()))
[  1   9  35  91 189 341 559]
[  0   0   2  18  60 140 270]
[  0   0   0   0   3  27  85]
[  0   0   0   0   0   0   4]
[  0   0   0   0   0   0   0]
```

Result

	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$
$\delta = 0$	9	35	91	189	341	559
$\delta = 1$	0	2	18	60	140	270
$\delta = 2$	0	0	0	3	27	85
$\delta = 3$	0	0	0	0	0	4
$\delta \geq 4$	0	0	0	0	0	0

Table: Number of sections of $\mathcal{O}(-\ell K_{\hat{D}} - \delta \hat{D})$.

Comparing with Tate's table, we can have at most a $SU(3)$ gauge group on the Enriques surface $\hat{D}$.

Landscape

Exercise

Repeat for all effective divisors in all toric 3-folds corresponding to reflexive polyhedra & find maximal possible gauge groups.

- Toric algorithm for effective cone.
- Discrete symmetries and overcounting.

Discrete Symmetries

The symmetries of the polytope ∇ must be modded out.

For example, $\hat{B} = \widehat{\mathbb{P}^3 / \mathbb{Z}_4}$:

```
sage: rays = Bsmooth.fan().rays()
sage: nabla = Polyhedron(vertices=rays)
sage: Aut = nabla.restricted_automorphism_group()
sage: Aut
Permutation Group with generators [(2,4)(6,7), (1,2)
    (3,4)(5,6)(7,8), (1,3)(5,8)]
sage: Aut.is_isomorphic( DihedralGroup(4) )
True
```