# Graph Theory using Sage

Chris Godsil

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# Outline

# Aims

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- I am keen to use sage as a tool for my research.
- I want my grad students to learn to make efficient use of it.
- I would very much like to use it as a tool in undergraduate combinatorics courses.

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- A second problem is competence. Most of my grad students know essentially nothing about combinatorial computing, and not much about computing.

# Solutions?

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- Appropriate documentation will help with the competence. (Note that useful python documentation is lacking too.)

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# A Construction

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- The vertices of our graph are the elements of $V$; two vectors $u$ and $v$ are adjacent if $u \neq v$ and the line spanned by $v-u$ lies in one of the parallel classes in $S$.
- Our graph will have $q^2$ vertices and valency $(q^2-1)/2$. It is a Cayley graph for the additive group of $V$.

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- The problem is two determine when the graphs in distinct families are not isomorphic.

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- How large can a clique be?
- How many cliques of maximum size on a given vertex?
- On a given edge?

# Our Experience

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- Sage was definitely the right tool to generate the graphs we wanted.
- The networkx clique routine was too slow; my student moved to using cliquer.

# Outline

# Definitions

### Definition

A matching in a graph is a set of disjoint edges. We denote the number of matchings in $X$ of size $r$ by $p(X, r)$. The matching polynomial $\mu(X, t)$ is given by

$$\mu(X, t) := \sum_{r \geq 0} (-1)^{n-r} p(X, r) t^{n-2r}$$

where $n = |V(X)|$.

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For a forest, the matching polynomial coincides with the characteristic polynomial (of the adjacency matrix).

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- Compute the matching polynomial for individual graphs on up 20 vertices.

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- All known algorithms for computing this polynomial take exponential time.
- There are 274668 graphs on 9 vertices.
- My student used polynomials as his data structure, and hence got bogged down in maxima.

# Outline

1. Introduction
   - Background
2. Student Projects
   - Conference Graphs
   - The Matching Polynomial
3. My Projects
   - The 600-Cell
   - Walk-Regular Graphs
   - Spectra of Trees

# 120 Vectors in $\mathbb{R}^4$

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- To this 96 vectors add the 16 vectors $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$.
- Complete the set to one of size 120 by adding each of the two signings of the four standard basis vectors.

# A Graph

For us, the 600-cell is the graph with these 120 vectors as its vertices, where two vectors are adjacent if they lie at distance $1/f$. Equivalently it is the 1-skeleton of the convex polytope generated by these vectors.

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- I was able to settle all the questions I had about this graph, by computation and using the insight gained.
- Do not use a list comprehension with index $i$ if you want $i$ to continue to denote a quaternion.

# Outline

# Definitions

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Suppose $X$ is a graph with adjacency matrix $A$. We say $X$ is walk regular if the diagonal entries of $A^r$ are all equal, for any non-negative $r$.

# Examples

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- Products of walk-regular graphs: Cartesian, direct, strong.

# Finding Regular Graphs

There are two sources:

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- McKay's `geng` can also be used. It is less efficient for larger graphs ($v \geq 15$ say).

# Conclusions

- The smallest walk-regular graph that is not vertex transitive is on 12 vertices.

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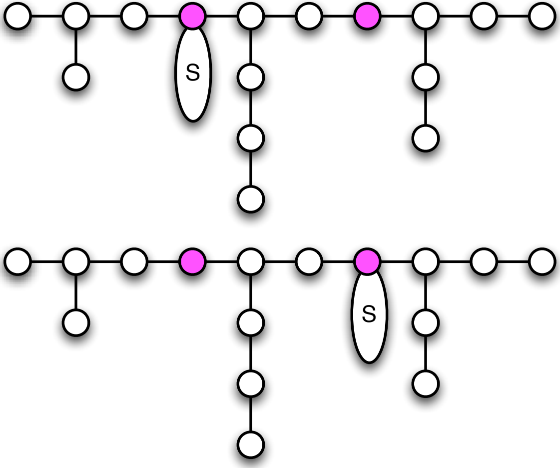
- Further progress will have to be made without computational help—there are too many graphs.

# Outline

# Cospectral Trees



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- Their Laplacians.
- Their line graphs.
- Their distance matrices.

# More Polynomials

We consider a two variable polynomial. Let $D$ denote the diagonal matrix of degrees of a graph and consider the polynomial:

$$\det(sI - tA + D).$$

# More Problems

- Sage defaults to an exponential algorithm for determinants. (I have worked around this, thanks to Mike Hansen.)

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- I will probably need to go to trees on 25 vertices :-(

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- Use a database as a tool for investigating graphs.
- Graph constructions using finite geometry

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